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$Q[\vec{x}] = (A\vec{x}, \vec{x}) \qquad A = A^*$

$D = S^* A S$ (this diagonalization is not unique)

what is unique?

r_+ - # of positive diagonal entries of D

r_- - # of negative " " " "

r_0 - # of zero " " " "

Theorem: Sylvester's law of inertia

r_+, r_-, r_0 do not depend on diagonalization

(Proof in text)

Explanation:

Def: A subspace E is called A-positive if $(A\vec{x}, \vec{x}) > 0$

$\forall \vec{x} \in E \quad \vec{x} \neq 0$

A subspace E is called negative if $(A\vec{x}, \vec{x}) < 0$

$\forall \vec{x} \in E \quad \vec{x} \neq 0$

A subspace E is called neutral if $(A\vec{x}, \vec{x}) = 0$

$\forall \vec{x} \in E$

r_+ is maximal dimension of positive subspaces

↳ we can find a positive subspace of at most dim r_+

r_- is m.d. of neg., r_0 is m.d. of neutral

Example:

$$\begin{array}{l} r_+ \\ \\ r_- \\ r_0 \end{array} \begin{pmatrix} 1 & & & & & & \\ & \ddots & & & & 0 & \\ & & 1 & & & & \\ & & & -1 & & & \\ & & & & \ddots & & \\ & & & & & -1 & \\ & 0 & & & & & 0 \\ & & & & & & & \ddots \\ & & & & & & & & 0 \end{pmatrix}$$

Span $(\vec{e}_1 \dots \vec{e}_{r_+})$ is a positive subspace

Type of Surface in $\mathbb{R}^n$ depends on number of positive and negative values

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$Q[\vec{x}] = (A\vec{x}, \vec{x})$

Def: 1) Q (A) is positive definite $^{(A>0)}$ if $(A\vec{x}, \vec{x}) > 0$ $\forall \vec{x} \neq 0$

2.) A is positive semi-definite $(A \geq 0)$ if $(A\vec{x}, \vec{x}) \geq 0$ $\forall \vec{x}$

3.) A is negative definite $(A < 0)$ if $(A\vec{x}, \vec{x}) < 0$ $\forall \vec{x} < 0$

4.) A is negative semidefinite $(A \leq 0)$ if $(A\vec{x}, \vec{x}) \leq 0$ $\forall \vec{x}$

5.) A is indefinite if $\exists\, \vec{x}_1, \vec{x}_2$ s.t. $(A\vec{x}_1, \vec{x}_1) > 0$, $(A\vec{x}_2, \vec{x}_2) < 0$

Theorem:

1.) A is $A > 0$ iff. all e.values > 0

2) $A \geq 0$ iff. all e.values ≥ 0

3) $A < 0$ iff. all e.values < 0

4.) $A \leq 0$ iff. all e.values ≤ 0

5.) A is indefinite iff A has positive and negative e.values

Ex.

① $A > 0, \quad B \geq 0 \Rightarrow A + B > 0$

$$((A+B)\vec{x}, \vec{x}) = \underbrace{(A\vec{x}, \vec{x})}_{>0} + \underbrace{(B\vec{x}, \vec{x})}_{\geq 0} > 0$$

② $A > 0 \Rightarrow A^{-1} > 0$

$A > 0 \Rightarrow$ all e.values λ_k of A are > 0

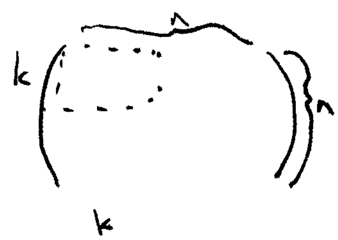
$\Rightarrow \frac{1}{\lambda_k}$ are e.values of A^{-1}

$\Rightarrow$ All e.values of A^{-1} are > 0

$\Rightarrow A^{-1} > 0$

Finding positive / negative definiteness

Theorem: Sylvester's criterion of positivity

$A_{n \times n}$ $\quad A_k =$ 

$A > 0$ iff $\det A_k > 0 \quad \forall k = 1, 2, \ldots n$

Example: $\begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$

$\det A_1 = \det 2 = 2$

$\det A_2 = 2(3) - 2(2) = 2$

$A > 0$

"Word of caution"!

$A < 0$ iff. $\det A_k < 0 \quad \forall k = 1, 2, \ldots, n$? WRONG!

$A < 0$ iff $-A > 0 \Leftrightarrow (-1)^k \underbrace{\det A_k}_{\det(-A_k)} > 0$ $\quad \det A_1 < 0, \det A_2 >$

determinants alternate