Review problems for MA 54, spring 2004.

- Below are the review problems for the final. They are mostly homework problems, or very similar.
- If you are comfortable doing these problems, you should be fine on the final.
- Please, do not assume that you will have the same (or even the same type of) problem on your final. Also, the distribution of problems is not indicative of the distribution of problems on the final between different topics.
- You should also review your midterms.

0.1 Linear Transformations, linear independence, rank theorem

1. For each linear transformation below find it matrix

   a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y)^T = (x + 2y, 2x - 5y, 7y)^T$;

   b) $T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by $T(x_1, x_2, x_3, x_4)^T = (x_1 + x_2 + x_3 + x_4, x_2 - x_4, x_1 + 3x_2 + 6x_4)^T$;

   c) $T : \mathbb{P}_n \to \mathbb{P}_n$, $Tf(t) = f'(t)$ (find the matrix with respect to the standard basis $1, t, t^2, \dots, t^n$);

   d) $T : \mathbb{P}_n \to \mathbb{P}_n$, $Tf(t) = 2f(t) + 3f'(t) - 4f''(t)$ (again with respect to the standard basis $1, t, t^2, \dots, t^n$).

2. Find the matrix of the reflection through the line $y = 3x$. You can leave your answer in the form of product.

3. Find linear transformations $A, B : \mathbb{R}^2 \to \mathbb{R}^2$ such that $AB = \mathbf{0}$ but $BA \neq \mathbf{0}$.

4. Find the matrix of the rotation in $\mathbb{R}^3$ by the angle α around the vector $(1, 2, 3)^T$. We assume that rotation is counterclockwise if we sit at the tip of the vector and looking at the origin.

   You can present the answer as a product of several matrices: you don't have to perform the multiplications.

5. Solve linear system

$$\begin{cases} x_1 + 2x_2 - x_3 + 3x_4 = 2 \\ 2x_1 + 4x_2 - x_3 + 6x_4 = 5 \\ x_2 + 2x_4 = 3 \end{cases}$$

6. For what value of b the system

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ b \end{pmatrix}$$

has a solution. Find the general solution of the system for this value of b.

7. Find the inverse of the matrix.

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 7 & 3 \\ 2 & 3 & 4 \end{pmatrix}$$

Show all steps

8. Suppose A is an $m \times n$ matrix of rank r. Under what conditions on these numbers

a) A is invertible?

b) The equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for every right side $\mathbf{b}$?

9. Prove or disprove: If the columns of a square $(n \times n)$ matrix A are linearly independent, so are the rows of $A^3 = AAA$.

10. Prove, that if E and F are subspaces of $\mathbb{R}^n$ and $\dim E + \dim F > n$, then $E \cap F \neq \{\mathbf{0}\}$. i.e. that there exists a vector $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{x} \in E \cap F$

11. For the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 8 & 3 \end{pmatrix}$$

find its rank and the dimensions of four fundamental subspaces.

Find bases in its fundamental subspaces.

12. A 54×37 matrix has rank 31. What are dimensions of all 4 fundamental subspaces?

0.2 Determinants

13. If A is an $n \times n$ matrix, how the determinants $\det A$ and $\det(5A)$ are related? **Remark:** $\det(5A) = 5 \det A$ only in the trivial case of 1×1 matrices

14. How the determinants $\det A$ and $\det B$ are related if

a)

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \qquad B = \begin{pmatrix} 2a_1 & 3a_2 & 5a_3 \\ 2b_1 & 3b_2 & 5b_3 \\ 2c_1 & 3c_2 & 5c_3 \end{pmatrix}.$$

b)

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}, \qquad B = \begin{pmatrix} 3a_1 & 4a_2 + 5a_1 & 5a_3 \\ 3b_1 & 4b_2 + 5b_1 & 5b_3 \\ 3c_1 & 4c_2 + 5c_1 & 5c_3 \end{pmatrix}.$$

15. Using column or row operations compute the determinant

$$\begin{vmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{vmatrix}.$$

16. Evaluate the determinant using any method

$$\begin{vmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{vmatrix}$$

17. Use row (column) expansion to evaluate the determinants). Note, that you don't need to use the first row (column): picking row (column) with many zeroes will simplify your calculations.

$$\begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 5 \\ 1 & -3 & 0 \end{vmatrix}, \qquad \begin{vmatrix} 4 & -6 & -4 & 4 \\ 2 & 1 & 0 & 0 \\ 0 & -3 & 1 & 3 \\ -2 & 2 & -3 & -5 \end{vmatrix}$$

18. True or false

a) Determinant is only defined for square matrices.

b) If two rows or columns of A are identical, then $\det A = 0$.

c) If B is the matrix obtained from A by interchanging two rows (or columns), then $\det B = \det A$.

d) If B is the matrix obtained from A by multiplying a row (column) of A by a scalar α, then $\det B = \det A$.

e) If B is the matrix obtained from A by adding a multiple of a row to some other row, then $\det B = \det A$.

f) The determinant of a triangular matrix is the product of its diagonal entries.

g) $\det(A^T) = -\det(A)$.

h) $\det(AB) = \det(A)\det(B)$.

i) A matrix A is invertible if and only if $\det A \neq 0$.

j) If A is an invertible matrix, then $\det(A^{-1}) = 1/\det(A)$.

19. Using cofactor formula compute inverses of the matrices

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

20. Why is there an even number of permutations of $(1, 2, \ldots, 9)$ and why are exactly half of them odd permutations? **Hint:** this problem can be hard to solve in terms of permutations, but there is a very simple solution using determinants.

21. Compute the determinant of

$$A_4 = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

Find also the determinants of smaller matrices A_3 and A_2 with the same pattern of zeroes on the diagonal and ones elsewhere. Can you predict $\det A_n$? Can you justify it? **Hint:** Look at the two problems below.

22. Let P be the orthogonal projection onto subspace E of an inner product space V, $\dim V = n$, $\dim E = r$. Find eigenvalues, eigenvectors (eigenspaces). Find algebraic and geometric multiplicities of each eigenvalue.

23. a) Find the matrix of the orthogonal projection onto one-dimensional subspace in $\mathbb{R}^n$ spanned by the vector $(1, 1, \ldots, 1)^T$.

b) Let A be the $n \times n$ matrix with all entries equal 1. Compute its eigenvalues and their multiplicities (use previous problem).

c) Compute eigenvalues (and multiplicities) of the matrix $A - I$, i.e. of the matrix with zeroes on the main diagonal and ones everywhere else.

d) Compute $\det(A - I)$.

0.3 Basic spectral theory

24. Let A be a 3×3 matrix with eigenvalues $1, 2, 5$. What is $\operatorname{trace}(A^3)$, $\det(A^2 + 2A - 2I)$?

25. Are the matrices

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 2 \\ 4 & 2 \end{pmatrix}$$

similar? Justify.

26. Let A be a 5×5 matrix with 3 eigenvalues (not counting multiplicities). Suppose we know that one eigenspace is three-dimensional. Can you say if A is diagonalizable?

27. Give an example of a 3×3 matrix which cannot be diagonalized. After you constructed the matrix, can you make it "generically looking", so no special structure of the matrix could be seen?

28. Let A be a 3×3 matrix with eigenvalues $1, 2, 5$. What is $\operatorname{trace}(A^3)$, $\det(A^2 + 2A - 2I)$?

29. a) Consider the transformation T in the space $M_{2\times 2}$ of 2×2 matrices, $T(A) = A^T$. Find all its eigenvalues and eigenvectors. Is it possible to diagonalize this transformation? **Hint:** While it is possible to write a matrix of this linear transformation in some basis, compute characteristic polynomial, and so on..., it is easier to find eigenvalues and eigenvectors directly from the definition.

b) Can you do the same problem but in the space of $n \times n$ matrices?

0.4 Orthogonal projections, Gram–Schmidt, etc

30. Fit a plane $z = a + bx + cy$ to four points $(1, 1, 3)$, $(0, 3, 6)$, $(2, 1, 5)$, $(0, 0, 0)$.
To do that

a) Find 4 equations with 3 unknowns a, b, c such that the plane pass through all 4 points (This system does not have to have a solution)

b) Find the least square solution of the system

31. Apply Gram–Schmidt orthogonalization to the system of vectors $(1, 2, -2)^T$, $(1, -1, 4)^T$, $(2, 1, 1)^T$

32. Suppose P is the orthogonal projection onto a subspace E, and Q is the orthogonal projection onto the orthogonal complement $E^\perp$.

a) What are $P + Q$ and PQ?

b) Show that $P - Q$ is its own inverse.

33. Find matrices of orthogonal projections onto all 4 fundamental subspaces of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}.$$

Note, that really you need only to compute 2 of the projections. If you pick appropriate 2, the other 2 are easy to obtain from them (recall, how the projections onto E and $E^\perp$ are related)

0.5 Unitary equivalence, orthogonal diagonalization, self-adjoint, unitary and normal matrices

34. Which of the following pairs of matrices are unitarily equivalent:

a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

c) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

d) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}$.

e) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Hint: It is easy to eliminate matrices that are not unitarily equivalent: remember, that unitarily equivalent matrices are similar, and trace, determinant and eigenvalues of similar matrices coincide.

Also, the Frobenius norm helps in eliminating non unitarily equivalent matrices.

And finally, matrix is unitarily equivalent to a diagonal one if and only if it has an orthogonal basis of eigenvectors.

35. Orthogonally diagonalize the matrix,

$$A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix},$$

i.e. represent it as $A = UDU^*$, where D is diagonal and U is unitary.

Among all square roots of A, i.e. among all matrices B such that $B^2 = A$, find one that has positive eigenvalues. You can leave B as a product.

36. Orthogonally diagonalize the matrix,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

i.e. represent it as $A = UDU^*$, where D is diagonal and U is unitary.

Note, that one of the eigenvalues has multiplicity 2, so you need to find and orthonormal basis in the corresponding eigenspace (using Gram-Schmidt, for example).

37. True or false:

a) Every unitary operator $U : X \to X$ is normal.

b) A matrix is unitary if and only if it is invertible.

c) If two matrices are unitarily equivalent, then they are also similar.

d) The sum of self-adjoint operators is self-adjoint.

e) The adjoint of a unitary operator is unitary

f) The adjoint of a normal operator is normal.

g) If all eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.

h) If all eigenvalues of a normal operator are 1, then the operator is identity.

i) A linear operator may preserve norm but not the inner product.

38. True or false: The sum of normal operators is normal? Justify your conclusion.

39. Can a non-zero normal operator N be *nilpotent*, i.e. satisfy $N^k = 0$ for some power k? Justify your answer.

40. Show that if a normal matrix P ($P \neq 0$, $P \neq I$) satisfies $P^2 = P$ then P is an orthogonal projection onto some subspace.

0.6 Singular Value Decomposition

41. Let A be an invertible matrix, and let $A = W\Sigma V^*$ be its singular value decomposition. Find a singular value decomposition for A^* and A^{-1}.

42. Find singular value decomposition $A = W\Sigma V^*$ where V and W are unitary matrices for the following matrices

a) $A = \begin{pmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{pmatrix}$

b) $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$ (better to find SVD for A^* first).

43. Let A be a normal operator, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues (counting multiplicities). Show that singular values of A are $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$.

44. Find singular values, norm and condition number of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

You can do this problem practically without any computations, if you use the previous problem and can answer the following questions:

a) What are singular values (eigenvalues) of an orthogonal projection P_E onto some subspace E?

b) What is the matrix of the orthogonal projection onto the subspace spanned by the vector $(1, 1, 1)^T$?

c) How the eigenvalues of the operators T and $aT + bI$, where a and b are scalars, are related?

Of course, you can also just honestly do the computations.

0.7 Quadratic forms

45. Diagonalize the quadratic form with the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Use two methods: completion of squares and row operations. Which one do you like betterr?

Can you say if the matrix A is positive definite or not?

46. Using Silvester's Criterion of Positivity check if the matrices

$$A = \begin{pmatrix} 4 & 2 & 1 \\ 2 & 3 & -1 \\ 1 & -1 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & -1 & 2 \\ -1 & 4 & -2 \\ 2 & -2 & 1 \end{pmatrix}$$

is positive definite or not.

Are the matrices $-A$, A^3 and A^{-1}, $A + B^{-1}$, $A + B$, $A - B$ positive definite?

47. True or false:

a) If A is positive definite, then A^5 is positive definite

b) If A is negative definite, then A^8 is negative definite

c) If A is negative definite, then A^{12} is positive definite.

d) If A is positive definite and B is negative semidefinite, then $A - B$ is positive definite

e) If A is indefinite, and B is positive definite, then $A + B$ is indefinite.

Justify your answers