Math. 54-02 **Spring 2004.** **Test 1.**

Name:
Student #

Problem	1	2	3	4	5	6	7	total
Max Possible	14	14	14	14	14	15	15	100
Score								

Work out the problems on the space provided. If more space is needed use back side of the pages or scratch paper.

To ensure maximal partial credit show all your work.

After you are done with the test, please comment on its difficulty:
The test was:
☐ Too easy ☐ Fairly easy ☐ Right on ☐ Challenging ☐ Unfairly demanding

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1. Let A and B be $n \times n$ matrices. True or false: if the product AB is invertible, then both A and B are invertible. Justify.

True $\det(AB) = \det A \cdot \det B$ (*)

AB invertible $\Rightarrow \det(AB) \neq 0$

(*) $\Rightarrow \det A \neq 0, \det B \neq 0$

$\Rightarrow A, B$ invertible

Another explanation: Let $C = AB$

$I = C^{-1}\underbrace{AB}_{C} = (C^{-1}A)B \Rightarrow C^{-1}A$ is left inverse for B

Therefore B is Left. Invert, so because it is $n \times n$ it is invertible

Similarly BC^{-1} is right inverse for A

2. True or false: If A is 9×12 matrix with 3-dimensional nullspace $\operatorname{Ker} A$, then $A\mathbf{x} = \mathbf{b}$ can be solved for every $\mathbf{b}$. Justify

True: rank $A = 12 - 3 = 9 \Rightarrow 9$ pivots

$\Rightarrow$ pivot in every row

$\Rightarrow A\vec{x} = \vec{b}$ has a solution $\forall \vec{b}$

3. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be vectors in $\mathbb{R}^n$. True or false: if the system $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is linearly independent, it is a basis. Justify.

True Let $A = [\vec{v}_1\ \vec{v}_2 \ldots \vec{v}_n]$ - matrix with columns $\vec{v}_1 \ldots \vec{v}_n$

$\vec{v}_1 \ldots \vec{v}_n$ LI $\Rightarrow$ ① pivot in every column

$\Rightarrow$ n pivots $\Rightarrow$ ② pivot in every row (A is $n \times n$)

①&② $\Rightarrow \vec{v}_1 \ldots \vec{v}_n$ is a basis

4. For the matrix

$$A = \begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 8 & 3 \end{pmatrix}$$

find its rank and the dimensions of four fundamental subspaces.

Find bases in its column space and row space.

$$\begin{array}{c} \\ +r_1 \\ -r_1 \end{array}\begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ -1 & -2 & 1 & 1 & 0 \\ 1 & 2 & 2 & 8 & 3 \end{pmatrix} \sim \begin{array}{c} \\ \\ -2r_2 \end{array}\begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 6 & 2 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 2 & 0 & 2 & 1 \\ 0 & 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\therefore$ $\dim \operatorname{Ran} A = \dim \operatorname{Ran} A^T = 2$ (rank = # of pivots)

$\therefore$ $\dim \operatorname{Ker} A = 5 - \operatorname{rank} A = 5 - 2 = 3$

$\therefore$ $\dim \operatorname{Ker} A^T = 3 - 2 = 1$

$\therefore$ Basis in $\operatorname{Ran} A$: $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ (columns # 1, 3 of A)

$\therefore$ Basis in $\operatorname{Ran} A^T$: $\begin{pmatrix} 1 \\ 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 1 \end{pmatrix}$ rows # 1, 3 of the echelon form

5. An $n \times n$ matrix A is called *skew symmetric* if $A^T = -A$. Prove that if A is skew symmetric and n is odd, then $\det A = 0$.

$\det(-A) = (-1)^n \det A = -\det A$ because n is odd

$\det(A^T) = \det A$.

Since $A^T = -A$ we get $\det A = -\det A$

It is possible only if $\det A = 0$

6. For what value of b the system

$$\begin{pmatrix} 1 & 2 & 2 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 4 \\ b \end{pmatrix}$$

has a solution. Find the general solution of the system for this value of b.

$$\begin{array}{c} \\ -2r_1 \\ -r_1 \end{array}\left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 2 & 4 & 6 & 4 \\ 1 & 2 & 3 & b \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & b-1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & b-1 \end{array}\right)\begin{array}{c} \\ \\ -r_2 \end{array} \sim$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & b-2 \end{array}\right)\begin{array}{c} -2r_2 \\ \\ \\ \end{array} \sim \left(\begin{array}{ccc|c} \boxed{1} & 2 & 0 & -1 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ $\boxed{\text{Solution exists only for } b = 2}$

Continuing with $b = 2$

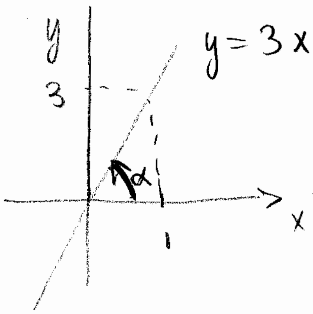
$x_3 = 1$

x_2 - free

$x_1 = -1 - 2x_2$

$$\begin{pmatrix} -1-2x_2 \\ x_2 \\ 1 \end{pmatrix} = \underbrace{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}}$$ general solution.

7. Find the matrix of the reflection through the line $y = 3x$. You can leave your answer in the form of product.



$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = T_0 \quad \text{- reflection in x-axis}$$

$$T = \underset{\substack{\text{Rot} \\ \text{through } \alpha}}{R_\alpha} \, T_0 \, \underset{\text{Rot through } \alpha}{R_{-\alpha}}$$

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

$$\cos\alpha = \frac{1}{\sqrt{3^2+1^2}} = \frac{1}{\sqrt{10}}$$

$$\sin\alpha = \frac{3}{\sqrt{10}}$$

$$R_\alpha = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \qquad R_{-\alpha} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$$

$$T = \frac{1}{10} \begin{pmatrix} 1 & -3 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -3 & 1 \end{pmatrix}$$