Homework assignment, Feb. 16, 2004.

1. True or false:

a) Every vector space that is generated by a finite set has a basis;

b) Every vector space has a (finite) basis;

c) A vector space cannot have more than one basis.

d) If a vector space has a finite basis, then the number of vectors in every basis is the same.

e) The dimension of $\mathbb{P}_n$ is n;

f) The dimension on $M_{m\times n}$ is $m+n$;

g) If vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ generate (span) the vector space V, then every vector in V can be written as a linear combination of vector $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in only one way.

h) Every subspace of a finite-dimensional space is finite-dimensional.

i) If V is a vector space having dimension n, then V has exactly one subspace of dimension 0 and exactly one subspace of dimension n.

j) If V is a vector space having dimension n, then a system of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is linearly independent if and only if it spans V.

2. (Problem from the first homework revisited: now this problem should be easy) Is it possible that vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent, but the vectors $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$, $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$ and $\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$ are linearly *independent*? **Hint:** What dimension the subspace $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ can have?

3. Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be a basis in V. Show that $\mathbf{u}+\mathbf{v}+\mathbf{w},\ \mathbf{v}+\mathbf{w},\ \mathbf{w}$ is also a basis in V.

4. True or false

a) Any system of linear equations has at least one solution.

b) Any system of linear equations has at most one solution.

c) Any homogeneous system of linear equations has at least one solution.

d) Any system of n linear equations in n unknowns has at least one solution.

e) Any system of n linear equations in n unknowns has at most one solution.

f) If the homogeneous system corresponding to a given system of a linear equations has a solution, then the given system has a solution.

g) If the coefficient matrix of a homogeneous system of n linear equations in n unknowns is invertible, then the system has no non-zero solution.

h) The solution set of any system of m equations in n unknowns is a subspace in $\mathbb{R}^n$.

i) The solution set of any homogeneous system of m equations in n unknowns is a subspace in $\mathbb{R}^n$.

5. What is the smallest subspace of the space of 4×4 matrices which contains all upper triangular matrices ($a_{j,k} = 0$ for all $j > k$), and all symmetric matrices ($A = A^T$)? What is the largest subspace contained in both of these subspaces?

6. Find inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 0 & 0 & 0 & 0 \\ 3 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 3 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 3 & 4 \end{pmatrix}$$

7. Find a 2×3 system (2 equations with 3 unknowns) such that its general solution has a form

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + s \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \qquad s \in \mathbb{R}.$$