Homework assignment, Feb. 18, 2004.

1. True or false:

   a) The rank of a matrix equal to the number of its non-zero columns.

   b) The $m \times n$ zero matrix is the only $m \times n$ matrix having rank 0.

   c) Elementary row operations preserve rank.

   d) Elementary column operations do not necessarily preserve rank.

   e) The rank of a matrix is equal to the maximum number of linearly independent columns in the matrix.

   f) The rank of a matrix is equal to the maximum number of linearly independent rows in the matrix.

   g) The rank of an $n \times n$ matrix is at most n.

   h) An $n \times n$ matrix having rank n is invertible.

2. A 54×37 matrix has rank 31. What are dimensions of all 4 fundamental subspaces?

3. Compute rank and find bases of all four fundamental subspaces for the matrices

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

4. Prove that if $A : X \to Y$ and V is a subspace of X then $\dim AV \le \operatorname{rank} A$. ($AV$ here means the subspace V transformed by the transformation A, i.e. any vector in AV can be represented as $A\mathbf{v}$, $\mathbf{v} \in V$). Deduce from here that $\operatorname{rank}(AB) \le \operatorname{rank} A$.

Remark: Here one can use the fact that if $V \subset W$ then $\dim V \le \dim W$. Do you understand why is it true?

5. Prove that if $A : X \to Y$ and V is a subspace of X then $\dim AV \le \dim V$. Deduce from here that $\operatorname{rank}(AB) \le \operatorname{rank} B$.

6. Prove that if the product AB of two $n \times n$ matrices is invertible, then both A and B are invertible. Even if you know about determinants, do not use them, we did not cover them yet. **Hint:** use previous 2 problems.