Homework assignment, Feb. 20, 2004.

1. Suppose A is an $m \times n$ matrix of rank r. Under what conditions on these numbers

   a) A is invertible?

   b) The equation $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for every right side $\mathbf{b}$?

2. Prove that if $A\mathbf{x} = \mathbf{0}$ has unique solution, then the equation $A^T\mathbf{x} = \mathbf{b}$ has a solution for every right side $\mathbf{b}$.
Hint: count pivots

3. Let A be a 22×40 matrix of rank 13. Find the dimensions each of the 4 fundamental subspaces.

4. Write a matrix with the required property, or explain why no such matrix exist

   a) Column space contains $(1,0,0)^T$, $(0,0,1)^T$, row space contains $(1,1)^T$, $(1,2)^T$.

   b) Column space is spanned by $(1,1,1)^T$, nullspace is spanned by $(1,2,3)^T$.

   c) Column space is $\mathbb{R}^4$, row space is $\mathbb{R}^3$.

Hint: Check first if the dimensions add up.

5. If A has the same four fundamental subspaces as B, does $A = B$?

6. Prove or disprove: If the columns of a square ($n \times n$) matrix A are linearly independent, so are the columns of $A^2 = AA$.
Remark: This is a review problem, it has nothing to do with rank. Recall how invertibility, basis and pivots are related.

7. Find rank of the matrix, and find the bases in all four fundamental subspaces

$$\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix}$$

8. Is there a matrix, whose nullspace and row space both contain the vector $(1,2,3)^T$. Justify (give an example of such matrix, or prove that it is impossible).

9. Joe Lucky had been assigned a homework problem to find a basis in the column space (range) of a 4×6 matrix. He was able to find out that that the rank of a matrix is 4, but then he got distracted and a dog ate his homework. Without knowing the matrix (but knowing that the rank is 4), can you help him and write a correct answer? Justify.