

## Homework assignment, March 10, 2004.

1. Explain why a  $5 \times 5$  matrix

$$\begin{pmatrix} * & * & * & * & * \\ * & * & * & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \\ 0 & 0 & 0 & * & * \end{pmatrix}$$

with a  $3 \times 3$  zero submatrix has determinant 0, regardless of the entries marked by  $*$ .

2. Give a counterexample to  $\det(A + B) = \det A + \det B$ . For what size of matrices the formula is true?

3. Where can you put zeroes into a  $4 \times 4$  matrix using as few as possible to guarantee that the determinant is zero?

Do not prove that you use the least possible number of zeroes, just give your best shot.

4. Where can you put zeroes and ones into a  $4 \times 4$  matrix using as few as possible to guarantee that the determinant is one?

Do not prove that you use the least possible number of zeroes and ones, just give your best shot.

5. Compute the determinants

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 3 & 1 \\ 1 & 4 & 1 & 1 \end{vmatrix}, \quad \begin{vmatrix} 9 & 1 & 9 & 9 & 9 \\ 9 & 0 & 9 & 9 & 2 \\ 4 & 0 & 0 & 5 & 0 \\ 7 & 0 & 3 & 9 & 0 \\ 5 & 0 & 0 & 7 & 0 \end{vmatrix}$$

6. Let  $\mathbf{v}_1, \mathbf{v}_2$  be vectors in  $\mathbb{R}^2$  and let  $A$  be the  $2 \times 2$  matrix with columns  $\mathbf{v}_1, \mathbf{v}_2$ . Prove that  $|\det A|$  is the area of the parallelogram with two sides given by the vectors  $\mathbf{v}_1, \mathbf{v}_2$ .

Consider first the case when  $\mathbf{v}_1 = (x_1, 0)^T$ . To treat general case  $\mathbf{v}_1 = (x_1, y_1)^T$  left multiply  $A$  by a rotation matrix that transforms vector  $\mathbf{v}_1$  into  $(\tilde{x}_1, 0)^T$ . **Hint:** what is the determinant of a rotation matrix?

7. Let  $A, B, C, D$  be  $n \times n$  matrices and let  $A$  be invertible. Using elementary “block row operations” show that

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - ACA^{-1}B) = \det(AD - ACA^{-1}B).$$

This formula is a bit more complicated than one for matrices with scalar entries.

You can use the fact that for a block triangular matrix

$$\det \begin{pmatrix} A & B \\ \mathbf{0} & D \end{pmatrix} = \det(A) \det(D).$$

Row operations can be realized as left multiplications by elementary block matrices, and the order of multiplication is essential.