Homework assignment, March 17, 2004.

1. Consider matrix

$$A = \begin{pmatrix} 2 & 6 & -6 \\ 0 & 5 & -2 \\ 0 & 0 & 4 \end{pmatrix}$$

a) Find its eigenvalues. Is it possible to find eigenvalues without computing?

b) Is this matrix diagonalizable? Find out without computing anything.

c) If the matrix is diagonalizable, diagonalize it.

2. Prove that eigenvalues (counting multiplicities) of a triangular matrix coincide with its diagonal entries

3. Diagonalize the matrix

$$\begin{pmatrix} 2 & 0 & 6 \\ 0 & 2 & 4 \\ 0 & 0 & 4 \end{pmatrix}.$$

4. Let A be a 5×5 matrix with 3 eigenvalues (not counting multiplicities). Suppose we know that one eigenspace is three-dimensional. Can you say if A is diagonalizable?

5. Let A be a square matrix with real entries, and let λ be its complex eigenvalue. Suppose $\mathbf{v} = (v_1, v_2, \dots, v_n)^T$ is a corresponding eigenvector, $A\mathbf{v} = \lambda\mathbf{v}$. Prove that the $\overline{\lambda}$ is an eigenvalue of A and $A\overline{\mathbf{v}} = \overline{\lambda}\overline{\mathbf{v}}$. Here $\overline{\mathbf{v}}$ is the complex conjugate of the vector $\mathbf{v}$, $\overline{\mathbf{v}} := (\overline{v}_1, \overline{v}_2, \dots, \overline{v}_n)^T$.

6. Give an example of a 3×3 matrix which cannot be diagonalized. After you constructed the matrix, can you make it "generically looking", so no special structure of the matrix could be seen?

7. Let us recall that the famous Fibonacci sequence:

$$0, 1, 1, 2, 3, 5, 8, 13, 21, \dots$$

is defined as follows: we put $\varphi_0 = 0$, $\varphi_1 = 1$ and define

$$\varphi_{n+2} = \varphi_{n+1} + \varphi_n.$$

We want to find a formula for φ_n. To do this

a) Find a 2×2 matrix A such that

$$\begin{pmatrix} \varphi_{n+2} \\ \varphi_{n+1} \end{pmatrix} = A \begin{pmatrix} \varphi_{n+1} \\ \varphi_n \end{pmatrix}$$

Hint: Add trivial equation $\varphi_{n+1} = \varphi_{n+1}$ to the Fibonacci relation $\varphi_{n+2} = \varphi_{n+1} + \varphi_n$.

b) Diagonalize A and find a formula for A^n.

c) Noticing that

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi_n \end{pmatrix} = A^n \begin{pmatrix} \varphi_1 \\ \varphi_0 \end{pmatrix} = A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

find a formula for φ_n. (You will need to compute an inverse and perform multiplication here).

d) Show that the vector $(\varphi_{n+1}/\varphi_n, 1)^T$ converges to an eigenvector of A.

What do you think, is it a coincidence?