Homework assignment, March 22, 2004.

1. Let $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 3$, $(\mathbf{u}, \mathbf{v}) = 2 + i$. Compute

$$\|\mathbf{u} + \mathbf{v}\|^2, \qquad \|\mathbf{u} - \mathbf{v}\|^2, \qquad (\mathbf{u} + \mathbf{v}, \mathbf{u} - i\mathbf{v}), \qquad (\mathbf{u} + 3i\mathbf{v}, 4i\mathbf{u}).$$

2. Find the set of all vectors in $\mathbb{R}^4$ orthogonal to vectors $(1, 1, 1, 1)^T$ and $(1, 2, 3, 4)^T$.

3. Let A be a real $m \times n$ matrix. Describe the set of all vectors orthogonal to $\operatorname{Ran} A^T$, and the set of all vectors orthogonal to $\operatorname{Ran} A$

4 *(Equality in Cauchy–Schwarz inequality).* Prove that

$$|(\mathbf{x}, \mathbf{y})| = \|\mathbf{x}\| \cdot \|\mathbf{y}\|$$

if and only if one of the vectors is a multiple of the other. **Hint:** analyze the proof of Cauchy–Schwarz inequality.

5. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be an orthonormal basis in V.

a) Prove that for any $\mathbf{x} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k$, $\mathbf{y} = \sum_{k=1}^{\infty} \beta_k \mathbf{v}_k$

$$(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{n} \alpha_k \overline{\beta}_k.$$

b) Deduce from this the *Parseval's identity*

$$(\mathbf{x}, \mathbf{y}) = \sum_{k=1}^{n} (\mathbf{x}, \mathbf{v}_k)\overline{(\mathbf{y}, \mathbf{v}_k)}$$

c) Assume now that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is only orthogonal basis, not orthonormal. Can you write down the Parseval's identity in this case?

6. Let V_1 and V_2 be orthogonal subspaces, $V_1 \perp V_2$ (that means any vector in V_1 is orthogonal to any vector in V_2). Prove that the only vector they have in common is zero vector, i.e. prove that $V_1 \cap V_2 = \{\mathbf{0}\}$.