Homework assignment, March 3, 2004.

1. True or false

   a) Determinant is only defined for square matrices.

   b) If two rows or columns of A are identical, then $\det A = 0$.

   c) If B is the matrix obtained from A by interchanging two rows (or columns), then $\det B = \det A$.

   d) If B is the matrix obtained from A by multiplying a row (column) of A by a scalar α, then $\det B = \det A$.

   e) If B is the matrix obtained from A by adding a mutiple of a row to some other row, then $\det B = \det A$.

   f) The determinant of a triangular matrix is the product of its diagonal entries.

   g) $\det(A^T) = -\det(A)$.

   h) $\det(AB) = \det(A)\det(B)$.

   i) A matrix A is invertible if and only if $\det A \neq 0$.

   j) If A is an invertible matrix, then $\det(A^{-1}) = 1/\det(A)$.

2. Let A be an $n \times n$ matrix. How are $\det(3A)$, $\det(-A)$ and $\det(A^2)$ related to $\det A$.

3. Using cofactor formula compute inverses of the matrices

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \qquad \begin{pmatrix} 19 & -17 \\ 3 & -2 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 0 \\ 3 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 0 & 1 & 1 \end{pmatrix}$$

4. Let D_n be the determinant of the $n \times n$ tridiagonal matrix

$$\begin{pmatrix} 1 & -1 & & & 0 \\ 1 & 1 & -1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 1 & -1 \\ 0 & & & 1 & 1 \end{pmatrix}$$

Using cofactor expansion show that $D_n = D_{n-1} + D_{n-2}$. This yields that the sequence D_n is the Fibonacci sequence $1, 2, 3, 5, 8, 13, 21, \ldots$

5. Vandermonde determinant revisited. Our goal is to prove the formula

$$\begin{vmatrix} 1 & c_0 & c_0^2 & \dots & c_0^n \\ 1 & c_1 & c_1^2 & \dots & c_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & c_n & c_n^2 & \dots & c_n^n \end{vmatrix} = \prod_{0 \le j < k \le n} (c_k - c_j)$$

for the $(n+1) \times (n+1)$ Vandermonde determinant.

We will aply induction. To do this

a) Check that the formula holds for $n = 1$, $n = 2$ (see the previous assignments).

b) Call the variable c_n in the last row x, and show that the determinant is a polynomial of degree n, $A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n$, with the coefficients A_k depending on $c_0, c_1, \ldots, c_{n-1}$.

c) Show that the polynomial has zeroes at $x = c_0, c_1, \ldots, c_{n-1}$, so it can be represented as $A_n \cdot (x - c_0)(x - c_1) \ldots (x - c_n)$, where A_n as above.

d) Assuming that the formula for the Vandermonde determinant is true for $n-1$, compute A_n and prove the formula for n.

6. Let points A, B and C in the plane $\mathbb{R}^2$ have coordinates (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively. Show that the area of triangle ABC is the absolute value of

$$\frac{1}{2} \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

Hint: use row operations and geometric interpretation of 2×2 determinants (area).

The following problem illustrates the power of block matrix notation.

7. Let A be an $m \times n$ matrix and B be an $n \times m$ one. Prove that

$$\det \begin{pmatrix} 0 & A \\ -B & I \end{pmatrix} = \det(AB).$$

Hint: While it is possible to transform the matrix by row operations to a form where the determinant is easy to compute, the easiest way is to right multiply the matrix by $\begin{pmatrix} I & 0 \\ B & I \end{pmatrix}$.

You can use the fact that $\begin{vmatrix} C & 0 \\ 0 & I \end{vmatrix} = \det C$.