Homework assignment, April 12, 2004.

1. Orthogonally diagonalize the matrix,

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Find all square roots of A, i.e. find all matrices B such that $B^2 = A$.

Note, that all square roots of A are self-adjoint.

2. True or false: any self-adjoint matrix has a sel-adjoint square root. Justify.

3. True or false:

a) A product of two self-adjoint matrices is self-adjoint.

b) If A is self-adjoint, then A^k is self-adjoint.

Justify your conclusions

4. Let A be $m \times n$ matrix. Prove that

a) A^*A is self-adjoint.

b) All eigenvalues of A^*A are non-negative.

c) $A^*A + I$ is invertible.

5. Give a proof if the statement is true, or give a counterexample if it is false;

a) If $A = A^*$ then $A + iI$ is invertible.

b) If U is unitary, $U + \frac{3}{4}I$ is invertible

c) If a matrix A is real, $A - iI$ is invertible

6. Let U be a 2×2 orthogonal matrix with $\det U = 1$. Prove that U is a rotation matrix.

7. Let U be a 3×3 orthogonal matrix with $\det U = 1$. Prove that

a) 1 is an eigenvalue of U;

b) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthonormal basis, such that $U\mathbf{v}_1 = \mathbf{v}_1$ (remember, that 1 is an eigenvalue), then in this basis the matrix of U is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix},$$

where α is some angle.

Hint: Show, that since $\mathbf{v}_1$ is an eigenvector of U, all entries below 1 must be zero, and since $\mathbf{v}_1$ is also an eigenvector of U^* (why?), all entries right of 1 also must be zero. Then show that the lower right 2×2 matrix is an orthogonal one with determinant 1, and use the previous problem.