Homework assignment, April 14, 2004.

Due Friday, 4/16 (collected)

1. True or false:

   a) Every unitary operator $U : X \to X$ is normal.

   b) A matrix is unitary if and only if it is invertible.

   c) If two matrices are unitarily equivalent, then they are also similar.

   d) The sum of self-adjoint operators is self-adjoint.

   e) The adjoint of a unitary operator is unitary

   f) The adjoint of a normal operator is normal.

   g) If all eigenvalues of a linear operator are 1, then the operator must be unitary or orthogonal.

   h) If all eigenvalues of a normal operator are 1, then the operator is identity.

   i) A linear operator may preserve norm but not the inner product.

2. True or false: The sum of normal operators is normal? Justify your conclusion.

3. Orthogonally diagonalize the matrix,

$$A = \begin{pmatrix} 7 & 2 \\ 2 & 4 \end{pmatrix},$$

i.e. represent it as $A = UDU^*$, where D is diagonal and U is unitary.

Among all square roots of A, i.e. among all matrices B such that $B^2 = A$, find one that has positive eigenvectors. You can leave B as a product.

4. Orthogonally diagonalize the rotation matrix

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}.$$

where α is not a multiple of π. Note, that you will get complex eigenvalues in this case.

5. Orthogonally diagonalize the matrix

$$A = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}.$$

Hints: you will get real eigenvalues in this case. Also, the trigonometric identities $\sin 2x = 2\sin x \cos x$, $\sin^2 x = (1 - \cos 2x)/2$, $\cos^2 x = (1 + \cos 2x)/2$ (applied to $x = \alpha/2$) will help to simplify expressions for eigenvectors.

6. Can you describe the linear transformation with matrix A from the previous problem geometrically? Is has a very simple geometric interpretation.

7. Prove that a normal operator with unimodular eigenvalues (i.e. with all eigenvalues satisfying $|\lambda_k| = 1$) is unitary. **Hint:** consider diagonalization