Homework assignment, April 16, 2004.

1. For an operator

$$A = \begin{pmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{pmatrix},$$

find its modulus $|A| = (A^*A)^{1/2}$.

2. Orthogonally diagonalize the matrix,

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix},$$

i.e. represent it as $A = UDU^*$, where D is diagonal and U is unitary.

Note, that one of the eigenvalues has multiplicity 2, so you need to find and orthonormal basis in the corresponding eigenspace (using Gram-Schmidt, for example).

3. Complete the system of vectors $\mathbf{v}_1 = (1, 1, 1, 1)^T$, $\mathbf{v}_2 = (1, 2, -1, -2)^T$ to an orthogonal basis in $\mathbb{R}^4$, i.e. find two vectors $\mathbf{v}_3, \mathbf{v}_4$ such that the system $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ is an orthogonal basis.

Probably the simplest way is to find an orthogonal basis in the orthogonal complement of $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$

4. Let N be a normal operator. Prove that $\operatorname{Ran} N = \operatorname{Ran} N^*$ and $\ker N = \ker N^*$. Hint: you can use diagonalization.

5. An elementary rotation in $\mathbb{R}^3$ is a rotation about one of the coordinate axis.

Prove that a 3×3 orthogonal matrix with the determinant 1 can be represented as a product of three elementary rotation.

Hint: Multiplying the matrix from the left by elementary rotations, make it upper triagular. Which upper triangular matrices are orthogonal?

6. (Finding eigenvalues and eigenvectors of self-adjoint operators) Let $A = A^*$ be positive semidefinite operator (all eigenvalues are non-negative). Let λ be the largest eigenvalue, and let k be its multiplicity. Prove that

$$\frac{1}{\|A^n\|_{\mathcal{F}}} A^n \to \frac{1}{\sqrt{k}} P_E \qquad \text{as } n \to \infty$$

where P_E is the orthogonal peojection onto the eigenspace $E = \ker(A - \lambda I)$.

Here $\|A\|_{\mathcal{F}}$ stands for the Frobenius norm of the operator A, $\|A\|_{\mathcal{F}} = \operatorname{trace}(A^*A) = \sum_{j,k=1}^{n} |A_{j,k}|^2$.