# Homework assignment, April 28, 2004.

1. Find norms and condition numbers for the following matrices:

a) $A = \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix}$.

b) $A = \begin{pmatrix} 5 & 3 \\ -3 & 3 \end{pmatrix}$.

For the matrix $A$ from part a) present an example of the right side $\mathbf{b}$ and the error $\Delta\mathbf{b}$ such that

$$\frac{\|\Delta\mathbf{x}\|}{\|\mathbf{x}\|} = \|A\| \cdot \|A^{-1}\| \cdot \frac{\|\Delta\mathbf{b}\|}{\|\mathbf{b}\|};$$

here $A\mathbf{x} = \mathbf{b}$ and $A\Delta\mathbf{x} = \Delta\mathbf{b}$.

2. Let $A$ be a normal operator, and let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be its eigenvalues (counting multiplicities). Show that singular values of $A$ are $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|$.

3. Find singular values, norm and condition number of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

You can do this problem practically without any computations, if you use the previous problem and can answer the following questions:

a) What are singular values (eigenvalues) of an orthogonal projection $P_E$ onto some subspace $E$?

b) What is the matrix of the orthogonal projection onto the subspace spanned by the vector $(1, 1, 1)^T$?

c) How the eigenvalues of the operators $T$ and $aT + bI$, where $a$ and $b$ are scalars, are related?

Of course, you can also just honestly do the computations.

4. Show that the rank of a matrix is the number of its non-zero singular values (counting multiplicities).

5. Show that the operator norm of a matrix $A$ coincides with its Frobenius norm if and only if the matrix has rank one. **Hint:** The previous problem might help.

6. For the matrix $A$

$$A = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix}$$

describe the inverse image of the unit ball, i.e. the set of all $\mathbf{x} \in \mathbb{R}^2$ such that $\|A\mathbf{x}\| \leq 1$. Use singular value decomposition.