Homework assignment, April 30, 2004.

1. Find the matrix of the bilinear form L on $\mathbb{R}^3$,

$$L(\mathbf{x}, \mathbf{y}) = x_1y_1 + 2x_1y_2 + 14x_1y_3 - 5x_2y_1 + 2x_2y_2 - 3x_2y_3 + 8x_3y_1 + 19x_3y_2 - 2x_3y_3.$$

2. Define the bilenear form L on $\mathbb{R}^2$ by

$$L(\mathbf{x}, \mathbf{y}) = \det[\mathbf{x}, \mathbf{y}],$$

i.e. to compute $L(\mathbf{x}, \mathbf{y})$ we form a 2×2 matrix with columns $\mathbf{x}, \mathbf{y}$ and compute its determinant.
Find the matrix of L.

3. Find the matrix of the quadratic form Q on $\mathbb{R}^3$

$$Q[\mathbf{x}] = x_1^2 + 2x_1x_2 - 3x_1x_3 - 9x_2^2 + 6x_2x_3 + 13x_3^2$$

4. Diagonalize the quadratic form with the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

Use two methods: completion of squares and row operations. Which one do you like betterr?
Can you say if the matrix A is positive definite or not?

5. For the matrix A from the previous assignment

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

orthogonally diagonalize the corresponging quadratic form, i.e. find a diagonal matrix D and a unitary matrix U such that $D = U^*AU$.

6 *(Cayley–Hamilton Theorem for diagonalizable matrices).* The Cayley–Hamilton theorem states that if A is a square matrix, and $p(\lambda) = \det(A - \lambda I) = \sum_{k=0}^n c_k\lambda^k$ is its characteristic polynomial, them $p(A) := \sum_{k=0}^n c_kA^k = \mathbf{0}$ (we assuming, that by definition $A^0 = I$).

Prove this theorem for the special case when A is similar to a diagonal matrix, $A = SDS^{-1}$.

Hint: If $D = \text{Diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ and p is any polynomial, can you compute $p(D)$? What about $p(A)$?