Homework assignment, April 7, 2004.

1. Find matrices of orthogonal projections onto all 4 fundamental subspaces of the matrix

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 3 & 2 \\ 2 & 4 & 3 \end{pmatrix}.$$

Note, that really you need only to compute 2 of the projections. If you pick appropriate 2, the other 2 are easy to obtain from them (recall, how the projections onto E and $E^\perp$ are related)

2. Let A be an $m \times n$ matrix. Show that $\ker A = \ker(A^*A)$.

To do that you need to prove 2 inclusions, $\ker(A^*A) \subset \ker A$ and $\ker A \subset \ker(A^*A)$. One of the inclusions is trivial, for the other one use the fact that

$$\|A\mathbf{x}\|^2 = (A\mathbf{x}, A\mathbf{x}) = (A^*A\mathbf{x}, \mathbf{x}).$$

3. Use the equality $\ker A = \ker(A^*A)$ to prove that

a) $\operatorname{rank} A = \operatorname{rank}(A^*A)$

b) If $A\mathbf{x} = \mathbf{0}$ has only trivial solution, A is left invertible. (You can just write a formula for a left inverse)

4. Suppose, that for a matrix A the matrix A^*A is invertible, so the orthogonal projection onto $\operatorname{Ran} A$ is given by the formula $A(A^*A)^{-1}A^*$. Can you write formulas for the orthogonal projections onto other 3 fundamental subspaces ($\ker A$, $\ker A^*$, $\operatorname{Ran} A^*$)?

5. Let a matrix P be self-adjoint ($P^* = P$) and let $P^2 = P$. Show that P is the matrix of an orthogonal projection. **Hint:** consider the decomposition $\mathbf{x} = \mathbf{x}_1 + \mathbf{x}_2$, $\mathbf{x}_1 \in \operatorname{Ran} P$, $\mathbf{x}_2 \perp \operatorname{Ran} P$ and show that $P\mathbf{x}_1 = \mathbf{x}_1$, $P\mathbf{x}_2 = \mathbf{0}$. For one of the equalities you will need self-adjointness, for the other one the property $P^2 = P$.

6. Prove the polarization identities

$$(A\mathbf{x}, \mathbf{y}) = \frac{1}{4}\big[(A(\mathbf{x}+\mathbf{y}), \mathbf{x}+\mathbf{y}) - (A(\mathbf{x}-\mathbf{y}), \mathbf{x}-\mathbf{y})\big] \qquad \text{(real case)}$$

and

$$(A\mathbf{x}, \mathbf{y}) = \frac{1}{4}\sum_{\alpha=\pm1,\pm i} \alpha(A(\mathbf{x}+\alpha\mathbf{y}), \mathbf{x}+\alpha\mathbf{y}) \qquad \text{(complex case)}$$