Homework assignment, April 9, 2004.

1. Orthogonally diagonalize the following matrices,

$$\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & 2 & 2 \\ 2 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}$$

i.e. for each matrix A find a unitary matrix U and a diagonal matrix D such that $A = UDU^*$

2. True or false: a matrix is unitarily equivalent to a diagonal one if and only if it has an orthogonal basis of eigenvectors.

3. Show that a product of unitary (orthogonal) matrices is unitary (orthogonal) as well.

4. Let $U : X \to X$ be a linear transformation on a finite-dimensional inner product space. True or false:

a) If $\|U\mathbf{x}\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in X$, then U is unitary.

b) If $\|U\mathbf{e}_k\| = \|\mathbf{e}_k\|$, $k = 1, 2 \ldots, n$ for some orthonormal basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$, then U is unitary.

Justify your answers with a proof or a counterexample.

5. Let A and B be unitarily equivalent $n \times n$ matrices.

a) Prove that $\operatorname{trace}(A^*A) = \operatorname{trace}(B^*B)$.

b) Use a) to prove that

$$\sum_{j,k=1}^{n} |A_{j,k}|^2 = \sum_{j,k=1}^{n} |B_{j,k}|^2.$$

c) Use b) to prove that the matrices

$$\begin{pmatrix} 1 & 2 \\ 2 & i \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} i & 4 \\ 1 & 1 \end{pmatrix}$$

are not unitarily equivalent.

6. Which of the following pairs of matrices are unitarily equivalent:

a) $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

b) $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}$.

c) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

d) $\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & i \end{pmatrix}$.

e) $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$.

Hint: It is easy to eliminate matrices that are not unitarily equivalent: remember, that unitarily equivalent matrices are similar, and trace, determinant and eigenvalues of similar matrices coincide.

Also, the previous problem helps in eliminating non unitarily equivalent matrices.

And finally, matrix is unitarily equivalent to a diagonal one if and only if it has an orthogonal basis of eigenvectors.

7. Show that for a square matrix A the equality $\det(A^*) = \overline{\det(A)}$ holds.