## Homework assignment, Feb. 16, 2004, Solutions.

- 1. True or false:
  - a) Every vector space that is generated by a finite set has a basis; True
  - b) Every vector space has a (finite) basis; **False**: the space C([0,1]) or the space of all polynomials has no finite basis, only infinite ones.
  - c) A vector space cannot have more than one basis. False, columns of any invertible  $n \times n$  matrix form a basis in  $\mathbb{R}^n$ .
  - d) If a vector space has a finite basis, then the number of vectors in every basis is the same. **True**.
  - e) The dimension of  $\mathbb{P}_n$  is n; False, dim  $\mathbb{P}_n = n + 1$  (n + 1 coefficients determine a polynomial of degree n).
  - f) The dimension on  $M_{m \times n}$  is m + n; False, dim  $M_{m \times n} = mn$ .
  - g) If vectors  $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$  generate (span) the vector space V, then every vector in V can be written as a linear combination of vector  $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$  in only one way. False.
  - h) Every subspace of a finite-dimensional space is finite-dimensional. True
  - i) If V is a vector space having dimension n, then V has exactly one subspace of dimension 0 and exactly one subspace of dimension n. True, these are  $\{0\}$  and V.
  - j) If V is a vector space having dimension n, then a system of vectors  $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$  is linearly independent if and only if it spans V. **True**.

*Proof.* If V is  $\mathbb{R}^n$ , let A be the matrix with columns  $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ .

The vectors are linearly independent  $\updownarrow$ there is a pivot in every column  $\updownarrow$ there are *n* pivots  $\updownarrow$ there is a pivot in every row  $\updownarrow$ the system  $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$  is generating.

If V is not  $\mathbb{R}^n$  we fix some basis  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$  and transform V to  $\mathbb{R}^n$  by isomorphism

$$\mathbf{b}_1 \mapsto \mathbf{e}_1, \mathbf{b}_2 \mapsto \mathbf{e}_2, \dots, \mathbf{b}_n \mapsto \mathbf{e}_n.$$

Or, we could just say in the beginning of the proof the magic phrase: "Since any space of dimension n is isomorphic to  $\mathbb{R}^n$  we can assume without loss of generality that V is  $\mathbb{R}^{n}$ "

2. (Problem from the first homework revisited: now this problem should be easy) Is it possible that vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly dependent, but the vectors  $\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2$ ,  $\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{v}_3$  and  $\mathbf{w}_3 = \mathbf{v}_3 + \mathbf{v}_1$  are linearly *independent*? Hint: What dimension the subspace span( $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ ) can have?

No, it is impossible: If the vectors  $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$  are linearly dependent, then one of the vectors is a linear combination of two others. Therefore the subspace  $V := \operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$  is generated by these 2 vectors.

Since any generating system in V must consist of a least dim V vectors (Proposition 4.3), we can conclude that dim  $V \leq 2$ . Therefore, V cannot have 3 linearly independent vectors  $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$  in it.

3. Let vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  be a basis in V. Show that  $\mathbf{u} + \mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w}, \mathbf{w}$  is also a basis in V.

*Proof.* Let us transform the space V to  $\mathbb{R}^3$  by the isomorphism T that maps

 $\mathbf{u} \mapsto \mathbf{e}_1, \qquad \mathbf{v} \mapsto \mathbf{e}_2, \qquad \mathbf{w} \mapsto \mathbf{e}_3.$ 

The vectors  $\mathbf{u} + \mathbf{v} + \mathbf{w}$ ,  $\mathbf{v} + \mathbf{w}$ ,  $\mathbf{w}$  will be transformed of the vectors

$$(1,1,1)^T$$
,  $(0,1,1)^T$ ,  $(0,0,1)^T$ 

respectively. Putting the columns in the matrix (reversing the order of vectors) we get the matrix

$$\left(\begin{array}{rrr} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{array}\right)$$

whose echelon form can be obtained by interchanging the first and the last row. This echelon form has pivots in every row and column, so we have a basis.

Note, that if we move everything to  $\mathbb{R}^3$  by isomorphism that transforms  $\mathbf{u} \mapsto \mathbf{e}_3$ ,  $\mathbf{v} \mapsto \mathbf{e}_2$ ,  $\mathbf{w} \mapsto \mathbf{e}_1$ , we do not even need to interchange rows.

4. True or false

- a) Any system of linear equations has at least one solution. False, it can have 0, 1, or infinitely many solutions.
- b) Any system of linear equations has at most one solution. False, see above.
- c) Any homogeneous system of linear equations has at least one solution. True,  $\mathbf{x} = \mathbf{0}$  always solves  $A\mathbf{x} = \mathbf{0}$ .
- d) Any system of n linear equations in n unknowns has at least one solution. False, see (a)

- e) Any system of n linear equations in n unknowns has at most one solution. False, see (a)
- f) If the homogeneous system corresponding to a given system of a linear equations has a solution, then the given system has a solution. False, a homogeneous system always has a solution.
- g) If the coefficient matrix of a homogeneous system of n linear equations in n unknowns is invertible, then the system has no non-zero solution. True
- h) The solution set of any system of m equations in n unknowns is a subspace in  $\mathbb{R}^n$ . **False**. It *can* be a subspace, but it also cab be a translated subspace (affine manifold) or an empty set  $(\emptyset)$ .
- i) The solution set of any homogeneous system of m equations in n unknowns is a subspace in  $\mathbb{R}^n$ . True, the solution set is ker A.

5. What is the smallest subspace of the space of  $4 \times 4$  matrices which contains all upper triangular matrices  $(a_{j,k} = 0 \text{ for all } j > k)$ , and all symmetric matrices  $(A = A^T)$ ? What is the largest subspace contained in both of these subspaces?

Any square matrix can be represented as a sum of a symmetric matrix and of an upper triangular one (can you show how for  $4 \times 4$  matrices?) Therefore, any matrix must belong to a subspace containing all symmetric and all upper triangular matrices.

Therefore, the smallest subspace of the space of  $4 \times 4$  matrices which contains all upper triangular matrices  $(a_{j,k} = 0 \text{ for all } j > k)$ , and all symmetric matrices  $(A = A^T)$  is the whole space  $M_{4\times 4}$ .

For the second part, if a matrix is both upper triangular and symmetric, it must be diagonal. Indeed, since it is upper triangular, all entries below main diagonal are 0, and since it is symmetric, all entries above the main diagonal must be 0.

It is easy to check that collection of diagonal  $4 \times 4$  matrices is a subspace of  $\mathbb{R}^4$ . Therefore the largest subspace contained in both of these subspaces is the subspace of diagonal matrices.

6. Find inverse of the matrix

First of all notice that for

$$\begin{pmatrix}
1 & 2 & 0 & 0 & 0 & 0 \\
3 & 4 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 0 \\
0 & 0 & 3 & 4 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 3 & 4
\end{pmatrix}$$

$$A = \begin{pmatrix}
1 & 2 \\
3 & 4
\end{pmatrix}$$

$$A^{-1} = \begin{pmatrix}
-2 & 1 \\
3/2 & -1/2
\end{pmatrix}$$

,

we have

(I am omitting the process of inversion). The matrix we need to invert has block diagonal form

$$\left(\begin{array}{rrrr} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & A \end{array}\right)$$

where each block is a  $2 \times 2$  matrix, so its inverse is

$$\begin{pmatrix} A^{-1} & 0 & 0\\ 0 & A^{-1} & 0\\ 0 & 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0\\ 3/2 & -1/2 & 0 & 0 & 0\\ 0 & 0 & -2 & 1 & 0 & 0\\ 0 & 0 & 3/2 & -1/2 & 0 & 0\\ 0 & 0 & 0 & 0 & -2 & 1\\ 0 & 0 & 0 & 0 & 3/2 & -1/2 \end{pmatrix}$$

7. Find a  $2 \times 3$  system (2 equations with 3 unknowns) such that its general solution has a form

$$\begin{pmatrix} 1\\1\\0 \end{pmatrix} + s \begin{pmatrix} 1\\2\\1 \end{pmatrix}, \qquad s \in \mathbb{R}.$$

If we assume that variable s is free variable (say  $x_3$ ), we can write a solution as

$$\mathbf{x} = \left(\begin{array}{c} 1+x_3\\1+2x_3\\x_3\end{array}\right)$$

so we can just write the corresponding reduced echelon form of the augmented matrix:

$$\left(\begin{array}{rrrr|r} 1 & 0 & -1 & 1 \\ 0 & 1 & -2 & 1 \end{array}\right).$$

To make this system more "generically looking" one can perform some row operations on it.

One can also assume that s is the variable  $x_2$ , then the corresponding augmented matrix will be

$$\left(\begin{array}{rrrr|r} 1 & -1 & 0 & 1 \\ 0 & -2 & 1 & 1 \end{array}\right);$$

this matrix can be obtained from the above matrix by interchanging 3rd and 2nd column. It is not the reduced echelon form, but we can easily read the solution off it.