<u>HW 3/1</u> Solutions of # 6, # 5

6

$$\det(A+tI) = \begin{vmatrix} t & & & & & a_0 \\ -1 & t & & 0 & & a_1 \\ & -1 & t & & & a_2 \\ & & & \ddots & & \vdots \\ & 0 & & & \ddots & \\ & & & & -1 & a_{n-1}+t \end{vmatrix}$$

expanding in nth column

$$\overset{\downarrow}{=} (-1)^{n+1} a_0 \overbrace{\begin{vmatrix} -1 & t & & 0 \\ & -1 & t & \\ & & \ddots & \ddots \\ 0 & & & t \\ & & & -1 \end{vmatrix}}^{n-1} + (-1)^{n+2} a_1 \begin{vmatrix} t & & & 0 \\ -1 & t & & \\ & -1 & t & \\ & & \ddots & t \\ 0 & & & -1 \end{vmatrix}$$

$$+ (-1)^{n+1} a_2 \left|\begin{array}{cc|cccc} t & 0 & & & 0 & \\ -1 & t & & & & \\ \hline & & -1 & t & \ldots & 0 \\ & & & -1 & \ddots & t \\ & 0 & 0 & & & -1 \end{array}\right| + \ldots + (-1)^{n+n} (a_{n-1}+t) \cdot \begin{vmatrix} t & & & 0 \\ -1 & t & & \\ & -1 & \ddots & \\ 0 & & -1 & t \end{vmatrix}$$

block-diagonal

(*) see next page for term # k.

$$= (-1)^{n+1} \cdot a_0 \cdot (-1)^{n-1} + (-1)^{n+2} a_1 \, t \cdot (-1)^{n-2} + (-1)^{n+3} a_2 \, t^2 \cdot (-1)^{n-3}$$

$$+ (a_{n-1} + t) \cdot t^{n-1}$$

$$= a_0 + a_1 t + a_2 t^2 + \quad + a_{n-1} t^{n-1} + a_n t^n.$$

(*) term # $\underbrace{k}_{}^{(k=0,1,2\ldots n-1)}$ in the expansion =

$$(-1)^{n+k+1} a_k \left|\begin{array}{cccccc|c} t & & & & & & a_0 \\ -1 & t & & & & & a_1 \\ & -1 & & & & & \vdots \\ \hline & & & t & & & a_k \\ & & & -1 & t & & \\ & & & & -1 & \ddots & \\ & & & & & -1 & a_{n-1}+t \end{array}\right|$$

$$= (-1)^{n+k+1} a_k \left|\begin{array}{c|c} \begin{matrix} t & & & \\ -1 & t & & \\ & \ddots & \ddots & \\ & & -1 & t \end{matrix} & 0 \\ \hline 0 & \begin{matrix} -1 & t & & \\ & -1 & \ddots & \\ & & \ddots & t \\ & & & -1 \end{matrix} \\ \underbrace{\qquad\qquad}_{k} & \underbrace{\qquad\qquad}_{n-k-1} \end{array}\right|$$

$$= (-1)^{n+k} a_k \cdot t^k \cdot (-1)^{n-k-1} = a_k t^k$$

Alternative proof:

We want to show that $\det(A+tI) = a_0 + a_1 t + a_2 t^2 + \dots + a_{n-1} t^{n-1} + t^n$

Suppose the formula is true for $n-1$.

Then

$$\underbrace{\begin{vmatrix} t & \cdot & \cdot & \cdot & a_1 \\ -1 & t & \cdot & \cdot & a_2 \\ & -1 & t & 0 & \vdots \\ & & \ddots & \ddots & \\ 0 & & & -1 & a_{n-1}+t \end{vmatrix}}_{n-1} = a_1 + a_2 t + a_3 t^2 + \dots + a_{n-1} t^{n-2} + t^{n-1}$$

Induction hypothesis

Then (after checking for 1×1, 2×2 matrices)

$$\det(A+tI) = \begin{vmatrix} t & & & 0 & a_0 \\ -1 & t & & & a_1 \\ & -1 & t & & a_2 \\ & & \ddots & \ddots & \vdots \\ & 0 & & & a_{n-1}+t \end{vmatrix}$$

$$= t \cdot \begin{vmatrix} t & & & 0 & a_1 \\ -1 & t & & & a_2 \\ & -1 & \ddots & & \vdots \\ & 0 & & \ddots & \\ & & & & a_{n-1}+t \end{vmatrix} + (-1)^{n+1} a_0 \underbrace{\begin{vmatrix} -1 & t & & 0 \\ 0 & -1 & t & \\ & 0 & \ddots & \ddots \\ & & \ddots & t \\ 0 & & 0 & -1 \end{vmatrix}}_{n-1} \quad (*)$$

$$= t \cdot \underbrace{(a_1 + a_2 t + a_3 t^2 + \dots + a_{n-1} t^{n-2} + t^{n-1})}_{\text{induction hypothesis}} + (-1)^{n+1} a_0 (-1)^{n-1}$$

$$= a_0 + a_1 t + a_2 t^2 + \dots + a_{n-1} t^{n-1} + t^n$$

One more way:

Formula on the previous page can be written as:

$$\boxed{\begin{aligned} &D(a_0, a_1, \dots a_{n-1}; t) \qquad (*) \\ &= t \cdot D(a_1, a_2, \dots a_{n-1}; t) + a_0 \end{aligned}}$$

Where $D(c_0, c_1, \dots c_{n-1}; t)$ stands for

$$\begin{vmatrix} t & & & & 0 & c_0 \\ -1 & t & & & & c_1 \\ & -1 & t & & & c_2 \\ & & & \ddots & & \vdots \\ & 0 & & & & c_{n-1}+t \end{vmatrix}$$

We can check directly that

$D(a_{n-1}; t) = a_{n-1} + t$ (det. of 1×1 matrix)

$$D(a_{n-2}, a_{n-1}; t) = \begin{vmatrix} t & a_{n-2} \\ -1 & a_{n-1}+t \end{vmatrix} = a_{n-2} + a_{n-1}t + t^2$$

(note that (*) holds for these determinants)

Using (*) we get:

$$D(a_{n-3}, a_{n-2}, a_{n-1}; t) = a_{n-3} + a_{n-2}t + a_{n-1}t^2 + t^3$$

$\dots$

$$D(a_0, a_1, \dots, a_{n-1}, t) = a_0 + a_1 t + a_2 t^2 + \dots + a_{n-1}t^{n-1} + t^n;$$

#5

$$\begin{array}{r} \\ -r_1 \\ -r_2 \end{array}\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix}$$

$$= \begin{vmatrix} y-x & (y-x)(y+x) \\ z-x & (z-x)(z+x) \end{vmatrix} = (y-x)(z-x)\begin{vmatrix} 1 & y+x \\ 1 & \underbrace{z+x}_{-x\cdot \text{Col}_1} \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = (y-x)(z-x)(z-y)$$