Homework assignment, April 12, 2004.

1. Orthogonally diagonalize the matrix,

$$A = \begin{pmatrix} 3 & 2 \\ 2 & 3 \end{pmatrix}.$$

Find all square roots of A, i.e. find all matrices B such that $B^2 = A$.

Note, that all square roots of A are self-adjoint.

Solution: $A = UDU^*$ where

$$D = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix}, \qquad U = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

Square roots:

$$U \begin{pmatrix} \pm\sqrt{5} & 0 \\ 0 & \pm 1 \end{pmatrix} U^*$$

2. True or false: any self-adjoint matrix has a self-adjoint square root. Justify.

False, the square root of

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

are matrices

$$\begin{pmatrix} \pm i & 0 \\ 0 & \pm 1 \end{pmatrix}$$

none of which is self-adjoint

3. True or false:

a) A product of two self-adjoint matrices is self-adjoint.

False, consider the product

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}$$

b) If A is self-adjoint, then A^k is self-adjoint.

True, $(A^k)^* = (A^*)^k = A^k$

Justify your conclusions

4. Let A be $m \times n$ matrix. Prove that

a) A^*A is self-adjoint.

$(A^*A)^* = A^*A^{**} = A^*A$.

b) All eigenvalues of A^*A are non-negative.

Let $A\mathbf{x} = \lambda\mathbf{x}$. Then $(A^*A\mathbf{x}, \mathbf{x}) = (\lambda\mathbf{x}, \mathbf{x}) = \lambda\|\mathbf{x}\|^2$. On the other hand, $(A^*A\mathbf{x}, \mathbf{x}) = (A\mathbf{x}, A\mathbf{x}) = \|A\mathbf{x}\|^2 \geq 0$. So, $\lambda\|\mathbf{x}\|^2 \geq 0$ and therefore $\lambda \geq 0$ (note that $\|\mathbf{x}\| > 0$).

c) $A^*A + I$ is invertible. By the previous problem all eigenvalues must be non-negative, so -1 is not an eigenvalue.

5. Give a proof if the statement is true, or give a counterexample if it is false;

a) If $A = A^*$ then $A + iI$ is invertible.

b) If U is unitary, $U + \frac{3}{4}I$ is invertible

c) If a matrix A is real, $A - iI$ is invertible

6. Let U be a 2×2 orthogonal matrix with $\det U = 1$. Prove that U is a rotation matrix.

Let $\mathbf{u}_1$ be the first column of U. Since $\|\mathbf{u}_1\| = 1$ it can be written as $\mathbf{u} = (\cos\alpha, \sin\alpha)^T$ for some α. Any vector $\mathbf{x}$ orthogonal to $\mathbf{u}_1$ is a multiple of $(-\sin\alpha, \cos\alpha)^T$ (solve the equation $\mathbf{u}_1^*\mathbf{x} = \mathbf{0}$).

The second column $\mathbf{u}_2$ of U must be orthogonal to $\mathbf{u}_1$ and has unit norm. So, there are only two possibilities, $\mathbf{u}_2 = \pm(-\sin\alpha, \cos\alpha)^T$. But only $\mathbf{u}_2 = (-\sin\alpha, \cos\alpha)^T$ gives $\det U = 1$, the other choice gives $\det U = -1$.

7. Let U be a 3×3 orthogonal matrix with $\det U = 1$. Prove that

a) 1 is an eigenvalue of U;

b) If $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ is an orthonormal basis, such that $U\mathbf{v}_1 = \mathbf{v}_1$ (remember, that 1 is an eigenvalue), then in this basis the matrix of U is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix},$$

where α is some angle.

Hint: Show, that since $\mathbf{v}_1$ is an eigenvector of U, all entries below 1 must be zero, and since $\mathbf{v}_1$ is also an eigenvector of U^* (why?), all entries right of 1 also must be zero. Then show that the lower right 2×2 matrix is an orthogonal one with determinant 1, and use the previous problem.