Homework assignment, April 5, 2004.
Solutions

1. Find least square solution of the system

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Solution:

$$A^*A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad A^*\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Normal equation:

$$\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Solution:

$$\mathbf{x} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}$$

2. Find the matrix of the orthogonal projection P onto the column space of

$$\begin{pmatrix} 1 & 1 \\ 2 & -1 \\ -2 & 4 \end{pmatrix}.$$

Solution:

$$A^*A = \begin{pmatrix} 9 & -9 \\ -9 & 18 \end{pmatrix}, \qquad (A^*A)^{-1} = \begin{pmatrix} \frac{2}{9} & \frac{1}{9} \\ \frac{1}{9} & \frac{1}{9} \end{pmatrix}, \qquad P = A(A^*A)^{-1}A^* = \begin{pmatrix} \frac{5}{9} & \frac{4}{9} & \frac{2}{9} \\ \frac{4}{9} & \frac{5}{9} & -\frac{2}{9} \\ \frac{2}{9} & -\frac{2}{9} & \frac{8}{9} \end{pmatrix}$$

3. Find the best straight line fit (least square solution) to the points $(-2,4)$, $(-1,3)$, $(0,1)$, $(2,0)$.

Solution: Need to solve $ax_k + b = y_k$, $k = 1,2,3,4$ for a and b (x_k, y_k are given). Equation to solve by least squares:

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 1 \\ 0 \end{pmatrix}.$$

$$A^*A = \begin{pmatrix} 9 & -1 \\ -1 & 4 \end{pmatrix}, \qquad A^*\mathbf{b} = \begin{pmatrix} -11 \\ 8 \end{pmatrix}.$$

Normal equation $A^*A\mathbf{x} = A^*\mathbf{b}$:

$$\begin{pmatrix} 9 & -1 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -11 \\ 8 \end{pmatrix} \qquad \text{Solution: } \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} -\frac{36}{35} \\ \frac{61}{35} \end{pmatrix}$$

Answer: $y = -\frac{36}{35}x + \frac{61}{35}$.

4. Fit a plane $z = a + bx + cy$ to four points $(1, 1, 3)$, $(0, 3, 6)$, $(2, 1, 5)$, $(0, 0, 0)$. To do that

a) Find 4 equations with 3 unknowns a, b, c such that the plane pass through all 4 points (This system does not have to have a solution)

b) Find the least square solution of the system

Solution: Need to solve $a + bx_k + cy_k = z_k$, $k = 1, 2, 3, 4$ for a b and c (x_k, y_k, z_k are given). Equation to solve by least squares:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \\ 5 \\ 0 \end{pmatrix}.$$

$$A^*A = \begin{pmatrix} 4 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 11 \end{pmatrix}, \qquad A^*\mathbf{b} = \begin{pmatrix} 14 \\ 13 \\ 26 \end{pmatrix}.$$

Normal equation $A^*A\mathbf{x} = A^*\mathbf{b}$:

$$\begin{pmatrix} 4 & 3 & 5 \\ 3 & 5 & 3 \\ 5 & 3 & 11 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 14 \\ 13 \\ 26 \end{pmatrix} \qquad \text{Solution: } \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} -\frac{3}{25} \\ \frac{73}{50} \\ \frac{101}{50} \end{pmatrix}$$

Answer: $y = -0.75 + 1.46x + 2.02y$.

5. Suppose P is the orthogonal projection onto a subspace E, and Q is the orthogonal projection onto the orthogonal complement $E^\perp$.

a) Wnat are $P + Q$ and PQ?

b) Show that $P - Q$ is its own inverse.

Solution:

a) $P + Q = I$, $PQ = O$.

b) $(P - Q)(P - Q) =$ (using $PQ = QP = \mathbf{0}$) $= P^2 - QP - PQ + Q^2 = P^2 + Q^2 =$ (using $P^2 = P, Q^2 = Q$) $= P + Q = I$.