Solutions for the homework assignment, Feb. 6, 2004.

1. Find all right inverses to the 1×2 matrix (row) A = (1, 1). Conclude from here that the row A is not left invertible.

Solution: All right inverses of the matrix are $(x, 1 - x)^T$, $x \in \mathbb{R}$ (taking all possible values of x we get all possible right inverses).

The matrix is right invertible. If it is left invertible, it is invertible and its right inverse is unique (and coincide with the inverse). But we have more than one right inverse, so the matrix cannot be left invertible.

2. Find all left inverses of the column $(1,2,3)^T$

Solution: $(x, y, 1/3 - x/3 - 2x/3), x, y \in \mathbb{R}$.

3. Find two matrices A and B that AB is invertible, but A and B are not. **Hint:** square matrices A and B would not work. **Remark:** It is easy to construct such A and B in the case when the product AB is a 1×1 matrix (a scalar). But can you get 2×2 matrix AB? 3×3 ? $n \times n$?

Solution: Consider first 1×1 case. Let A = (1,0), and let $B = A^T$. Then AB = 1 (invertible 1×1 matrix, but

$$BA = \left(\begin{array}{cc} 1 & 0\\ 0 & 0 \end{array}\right)$$

and this matrix is not invertible.

For 2×2 case, let

$$A = \left(\begin{array}{rrr} 1 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right)$$

and let $B = A^T$. Then AB = I but

$$BA = \left(\begin{array}{rrrr} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0 \end{array}\right)$$

which is not invertible (a matrix with zero row or column cannot be invertible).

The pattern for 3×3 , $n \times n$ cases is clear.

4. Suppose the product AB is invertible. Show that A is right invertible and B is left invertible. **Hint:** you can just write formulas for right and left inverses.

Solution: Let C = AB. We know that C is invertible. Multiply equation AB = C by C^{-1} from the left. We get

$$ABC^{-1} = CC^{-1} = I,$$

i.e. BC^{-1} is a right inverse of A.

Another way to see that:

$$A(BC^{-1}) = (AB)C^{-1} = CC^{-1} = I$$

Similarly, $C^{-1}A$ is a left inverse for B.

5. Let A be $n \times n$ matrix. Prove that if $A^2 = \mathbf{0}$ then A is not invertible

Solution: If A is invertible, multiplying the equation $A^2 = \mathbf{0}$ by A^{-1} (does not matter, from the left or from the right) we get $A = \mathbf{0}$ which is a contradiction (**0** is not invertible).

6. Find a non-zero square matrix A such that $A^2 = \mathbf{0}$.

Solution:

$$\left(\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right)$$

7. Prove, that if $A: V \to W$ is an isomorphism (i.e. an invertible linear transformation) and $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis in V, then $A\mathbf{v}_1, A\mathbf{v}_2, \ldots, A\mathbf{v}_n$ is a basis in W.

Take arbitrary vector $\mathbf{w} \in W$. Since $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ is a basis in V, vector $A^{-1}\mathbf{w} \in V$ admits unique representation

$$A^{-1}\mathbf{w} = \sum_{k=1}^{n} \alpha_k \mathbf{v}_k.$$
 (1)

Applying transformation A to both sides of the above equality we get

$$\mathbf{w} = \sum_{k=1}^{n} \alpha_k A \mathbf{v}_k$$

Let us show that the above representation is unique. Suppose there is another representation

$$\mathbf{w} = \sum_{k=1}^{n} \widetilde{\alpha}_k A \mathbf{v}_k.$$

Applying transformation A^{-1} to this equality we get

$$A^{-}\mathbf{w} = \sum_{k=1}^{n} \widetilde{\alpha}_k \mathbf{v}_k.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ is a basis, such representation must be unique, so comparing with (1) we get that $\tilde{\alpha}_k = \alpha_k, k = 1, 2, \ldots, n$.

8. (Extra Credit) Find the matrix of the rotation in \mathbb{R}^3 by the angle α around the vector $(1, 2, 3)^T$. We assume that rotation is counterclockwise if we sit at the tip of the vector and looking at the origin.

You can present the answer as a product of several matrices: you don't have to perform the multiplications.