Math. 54-02 **Spring 2004.** **Test 2.**

Name:
Student #

Problem	1	2	3	4	5	6	7	8	total
Max Possible	12	13	12	13	13	12	13	12	100
Score									

Work out the problems on the space provided. If more space is needed use back side of the pages or scratch paper.

To ensure maximal partial credit show all your work.

After you are done with the test, please comment on its difficulty:
The test was:
☐ Too easy ☐ Fairly easy ☐ Right on ☐ Challenging ☐ Unfairly demanding

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1. Give an example of a square matrix, not similar to a diagonal one . Be sure to explain why it is not similar. Note, that the diagonal matrix is not required to be real, even if the original matrix has only real entries.

$A = \begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$ - e.value 0 of mult 2.
1 pivot, 1 free var, so geom. mult = 1

$2 \neq 1$ so ~~diag~~ cannot be diagonalized (alg. mult $\neq$ geom)

Another example: $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ e. value 2 of mult 2.

$2 \neq 1$ $A - 2I = \begin{pmatrix} 0 & \boxed{1} \\ 0 & 0 \end{pmatrix}$, dim ker $(A-2I) = 1$

2. Are the matrices

$$A = \begin{pmatrix} 1 & 3 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = B$$

similar or not? Are they unitarily equivalent? Justify.

You can answer both questions practically without any computations.

A has distinct e. values 1 & 2, so A is diagonalizable

$A = S \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} S^{-1}$ so A is similar to B

A is not unitarily equivalent to B $\|A\|_F \neq \|B\|_F$

$\sqrt{14}$ $\sqrt{5}$

3. Can a non-zero normal operator N be *nilpotent*, i.e. satisfy $N^k = 0$ for some power k? Justify your answer.

No. If N is normal, then its matrix in some ONB is $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$. Then $N^k = \mathrm{diag}(\lambda_1^k \dots \lambda_n^k)$. If $N^k = 0$, then $\lambda_j^k = 0 \ \forall j$ so all $\lambda_j = 0$. So $N = 0$.

4. Complete the system of vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{v}_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

to an orthogonal basis in $\mathbb{R}^4$. To do that you can find a basis in $\mathrm{span}\{\mathbf{v}_1, \mathbf{v}_2\}^\perp$ (this subspace is one of the fundamental subspaces) and make it orthogonal using Gram-Schmidt. Note, that you do not need to orthogonalize all 4 vectors.

$(\mathrm{span}\{\vec{v}_1\ \vec{v}_2\})^\perp$:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & 0 & -1/2 & 0 \\ 0 & \boxed{1} & 1/2 & 0 \end{pmatrix}$$

$x_1 = \frac{1}{2} x_3$

$x_2 = \frac{-1}{2} x_3$

x_3 - free

x_4 - free.

$$x = \begin{pmatrix} \frac{1}{2} x_3 \\ -\frac{1}{2} x_3 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1/2 \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

basis in $\mathrm{span}(\vec{x}_1\ \vec{x}_2)^\perp$

These vectors are already orthogonal, no need for Gram Schmidt

$$\therefore \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \\ -1/2 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

5. Show that if a self-adjoint matrix P ($P \neq 0$, $P \neq I$) satisfies $P^2 = P$ then P is an orthogonal projection onto some subspace.

$P = P^* \implies$ matrix of P is $\mathrm{diag}(\lambda_1, \lambda_2 \dots \lambda_n)$ in some ONB

$P^2 = P \implies \lambda_k^2 = \lambda_k \quad \forall k \quad$ so $\lambda_k = 1$ or $\lambda_k = 0$

so in some ONB matrix of P is

$\begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & 1 & & \\ & & & 0 & \\ 0 & & & & \ddots \; 0 \end{pmatrix}$ which means P is a projection

6. Find the best straight line fit (least squares) to the following measurements: $y = 2$ at $x = -1$; $\quad y = 0$ at $x = 0$; $\quad y = -3$ at $x = 1$; $\quad y = -5$ at $x = 2$.

$y_k = a x_k + b$

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix}}_{\vec{b}}$$

$$A^*A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$$

$\boxed{\text{Normal equation } A^*A\vec{x} = A^*\vec{b}}$

$$A^*\vec{b} = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \\ -5 \end{bmatrix} = \begin{bmatrix} -15 \\ -6 \end{bmatrix}$$

$$(A^*A)^{-1} = \frac{1}{24-4} \begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix} = \frac{1}{20} \begin{pmatrix} 4 & -2 \\ -2 & 6 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = (A^*A)^{-1} \vec{b} = \frac{1}{10} \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} -15 \\ -6 \end{pmatrix} = \frac{1}{10} \begin{bmatrix} -24 \\ -3 \end{bmatrix} = \begin{bmatrix} -2.4 \\ -0.3 \end{bmatrix}$$

$\boxed{y = -2.4x - 0.3}$ best fit

#5 Another proof:

Decompose $\vec{x} = \vec{x}_1 + \vec{x}_2$, $\vec{x}_1 \in \operatorname{Ran} P$, $\vec{x}_2 \perp \operatorname{Ran} P$

(Def. of orthogonal decomposition)

$\vec{x}_1 \in \operatorname{Ran} P$ means $\vec{x}_1 = P\vec{y}$ for some $\vec{y}$

$\Rightarrow P\vec{x}_1 = PP\vec{y} = P^2\vec{y} = P\vec{y} = \vec{x}$ (use $P^2 = 2$)

$\vec{x}_2 \perp \operatorname{Ran} P \Rightarrow \vec{x}_2 \in (\operatorname{Ran} P)^{\perp} = \operatorname{Ker} P^*$

$= \operatorname{Ker} P$

because $P = P^*$

Therefore $P\vec{x}_2 = \vec{0}$

So $P\vec{x} = P(\vec{x}_1 + \vec{x}_2) = P\vec{x}_1 + P\vec{x}_2 = \vec{x}_1 + \vec{0} = \vec{x}_1$.

That means P is the orthogonal projection onto $\operatorname{Ran} P$.

7. Find the matrix P of the orthogonal projection onto the subspace E spanned by the vector

$$\vec{V} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

and the matrix Q of the orthogonal projection onto $E^\perp$.

Compute $5P^3 + 3P^{17} + 6P^{94}$.

$$\therefore\ P = \frac{1}{\|\vec{V}\|^2} V \vec{V}^* = \frac{1}{14} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \end{pmatrix} = \underline{\frac{1}{14} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}},$$

$$\therefore\ Q = I - P = \underline{\frac{1}{14} \begin{pmatrix} 13 & -2 & -3 \\ -2 & 10 & -6 \\ -3 & -6 & 5 \end{pmatrix}}$$

Since P is projection, $P^2 = P$, so $P^k = P\ \forall k.$

$$5P^3 + 3P^{17} + 6P^{94} = 5P + 3P + 6P = 14P$$

$$= \underline{\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}}$$

8. Find a matrix with eigenvalues 1, 2, 3, and the corresponding eigenvectors

$$\vec{V_1} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix},\ \vec{V_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix},\ \vec{V_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

You can write you answer as a product.

Is such matrix unique? Can such matrix be normal?

$$\therefore\ A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 0 \end{pmatrix}^{-1}$$

$\therefore$ The matrix is unique: $\vec{V_1}\ \vec{V_2}\ \vec{V_3}$ is a basis and $A\vec{V_1} = \vec{V_1}$, $A\vec{V_2} = 2\vec{V_2}$, $A\vec{V_3} = 3\vec{V_3}$ and a linear transformation is determined by its action on a basis.

$\therefore$ A is not normal, because eigenvectors are not orthogonal $((\vec{V_1}, \vec{V_2}) = 5)$