Polygons with concurrent medians

John P. Steinberger
Department of Mathematics, UC Davis
jpsteinb@math.ucdavis.ca

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For Thomas Banchoff on the occasion of his 65th birthday.

Abstract

The *medians* of an odd-sided polygon are the lines joining vertices to the middle of their opposite sides. Triangles have concurrent medians but odd polygons with more sides are not always so lucky. In this paper we show that if $n-1$ medians of an odd n-gon are concurrent, then the remaining median is also concurrent with the others. Equivalently, for n odd any $\lceil \frac{n}{2} \rceil + 1$ consecutive vertices (in a general position) uniquely determine in every case an n-gon with concurrent medians. Thus pentagons with concurrent medians, for instance, are determined by choosing four vertices at random in the plane.

A short history

In the fall of 1998 I had the chance of being the teaching assistant for Thomas Banchoff's course *Fundamental problems of Geometry*. The course format relied heavily on a web-based discussion which I helped Tom monitor. One of the problems Tom put up for the students was simply the following: which pentagons have concurrent medians?

At that point Tom himself didn't even know the answer (if such a classification question ever has a "right" answer), which was typical of his mathematical *ouverture d'esprit*. However, Tom did tell me he supposed those pentagons were exactly the linear deformations of the regular pentagon. I agreed with him and gave it no more thought; the students would figure out the details for us.

Then a few days later I awoke sweaty and clammy-handed; I'd had a nightmare featuring a pentagon with concurrent medians and a pair of parallel sides. This

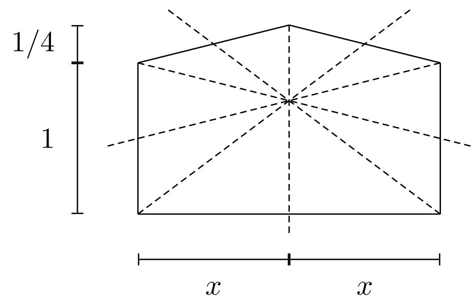

Figure 1: A pentagon with concurrent medians and parallel sides.

nightmarish pentagon (shown in Fig. 1) could not be the linear image of the regular pentagon since linear transformations preserve parallelism.

Without having seen the pentagon himself, Tom told me to show it in class. When I'd finished drawing the construction I had to run to meet another appointment, but with the pentagon still on the board and looking somewhat dismayed at the appearance of this odd case, Tom said: "You're making a very dramatic exit, do you realize?" It was undoubtedly, thanks to Tom, the high point of my career as a teaching assistant.

We thus knew the class of pentagons with concurrent medians was larger than the set of linear deformations of the regular pentagon, but without knowing exactly *how much* larger. Our initial observations seemed to indicate there were seven degrees of freedom in all: six degrees of freedom only accounted for all the translates of linear deformations of the regular pentagon (two degrees of freedom for the origin plus four degrees of freedom for the transformation), whereas eight degrees of freedom seemed too much. Indeed, from four vertices A, B, C and D—or a total of eight degrees of freedom—one could infer the point O where medians crossed, and from there a unique position for the last vertex E such that the line through O and B bisected the side DE (see Fig. 2), but there seemed no *a priori* reason why the line through O and C would bisect the side AE. In fact, our less-than-approximate hand sketches strongly suggested this wasn't the case.

Undeterred, one of the students, David Ziff, decided to model the construction on Geometer's Sketchpad. What he discovered was that the line through C and O *always* bisected the side AE. We were at once surprised and delighted by Ziff's discovery: it meant that we finally knew what had to be proved (naturally Geometer's Sketchpad was only a source of empirical evidence and it still befell to us to find a "real" proof).

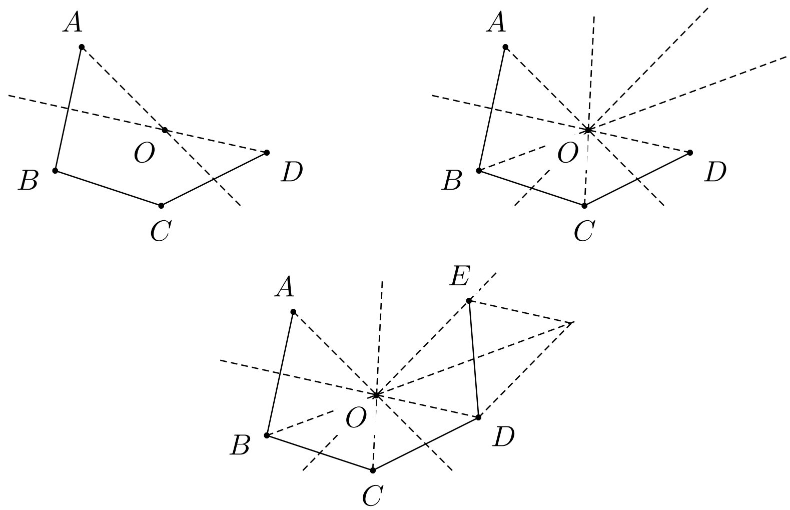

Figure 2: Determining the last vertex of a pentagon with concurrent medians from its first four vertices. It is not obvious whether A and E are equidistant from the line through O and C.

Banchoff found a first proof of Ziff's observation using vector geometry and I later devised a Euclidean proof (none of the students found a proof). Unknown to us at the time, G.C. Shephard had just recently mentioned the same result as a problem for the reader in a paper of his, the reference of which I have momentarily lost. I do not know any other references to this result. In this paper we shall give a new simple proof of Ziff's observation which applies not just to pentagons but to odd n-gons in general.

Our main result has two equivalent formulations:

Theorem 1 *If an odd n-gon has $n-1$ concurrent medians then the last median is concurrent with the rest.*

Theorem 2 *If n is odd and $P_1, \ldots, P_{\lceil n/2 \rceil +1}$ are points in general position then there exist unique points $P_{\lceil n/2 \rceil +2}, \ldots, P_n$ such that the polygon $P_1 P_2 \cdots P_n$ has concurrent medians.*

(A set of points is in "general position" if its distribution in the plane is essentially random.)

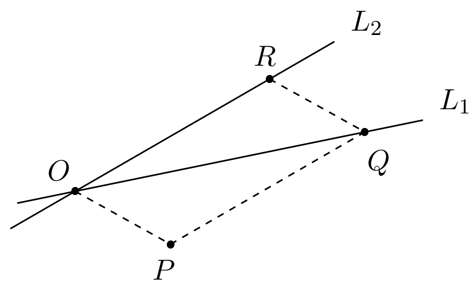

Figure 3

We shall first show that Theorem 2 is a consequence of Theorem 1. We shall then give a proof of Theorem 1 for the case $n = 5$, which the reader should have no problem generalizing to other values of n.

Notation: if A and B are two points then AB denotes the line through A and B, $|AB|$ denotes the 2-by-2 determinant of the points A and B considered as vectors, and $\overline{AB}$ denotes the segment from A to B. The median through a vertex V is always denoted M_V, even if the line has no label on the accompanying diagram (which we do to allow us to reduce clutter). If L is a line and S is a segment, then "L bisects S" and "L is a median of S" mean the same, namely that L contains the midpoint of S.

A reduction

In this section we show that Theorem 2 reduces to Theorem 1. It is sufficient to show that every group of $\lceil \frac{n}{2} \rceil + 1$ vertices uniquely determines an n-gon with $n - 1$ concurrent medians. We use the following elementary lemma:

Lemma 3 *(See Fig. 3) Let L_1 and L_2 be two non-parallel lines intersecting at a point O and let P be a point not on L_2. Let Q be the intersection of the parallel to L_2 through P with L_1 and let R be the intersection of L_2 with the parallel to OP through Q. Then R is the unique point on L_2 that is the same distance to L_1 as P and which is on the opposite side of L_1 from P.*

Proof: Obviously there exists only one point on L_2 which is the same distance to L_1 as P and which is on the other side of L_1 from P. But R is on L_2 and by construction P, Q, R, and O form a parallelogram whose diagonal OQ coincides with L_1, meaning

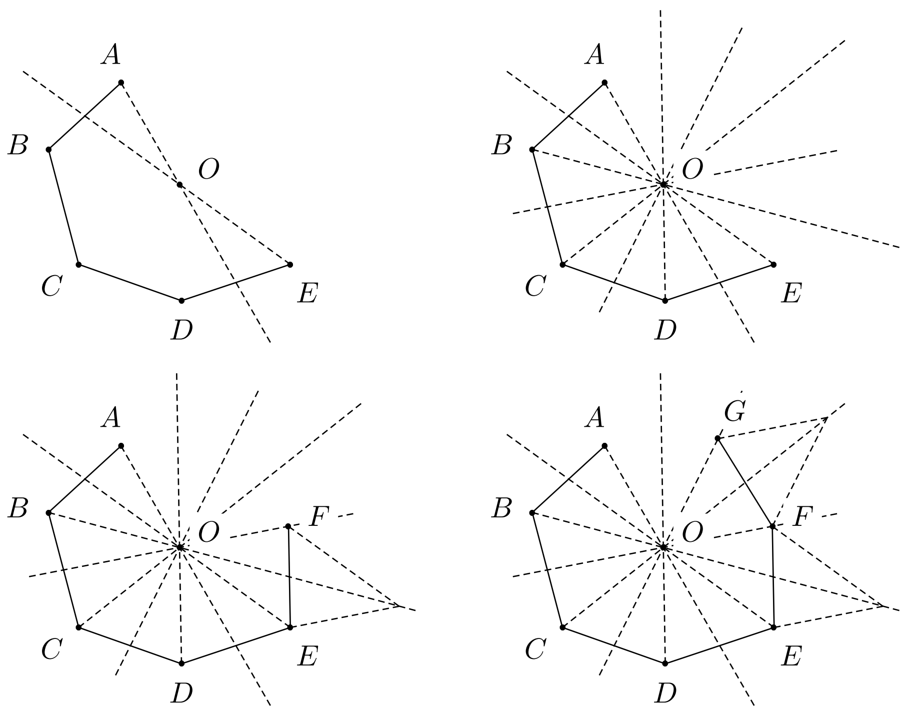

Figure 4

P and R are equidistant to L_1 and on opposite sides. ■

As a corollary the point E found in the construction of Fig. 2 is the unique point such that that the line through O and the midpoint of $\overline{BC}$ bisects the segment $\overline{DE}$. Thus when $n = 5$ any $4 = \lceil \frac{n}{2} \rceil + 1$ vertices in general position uniquely determine a pentagon with $4 = n - 1$ concurrent medians (namely the medians M_A, M_B, M_D and M_E). So Theorem 2 reduces to Theorem 1 when $n = 5$.

Now let $n = 7$. Since $\lceil \frac{n}{2} \rceil + 1 = 5$ we wish to show that any general placement of 5 consecutive vertices A, B, C, D and E uniquely determines a 7-gon with 6 concurrent medians. The point O of intersection of the medians is given by the intersection of medians M_A and M_E, which are known from the initial vertices. The rest of the construction proceeds analogously to the case $n = 5$ (see Fig. 4) until all vertices of the heptagon have been determined, at which point exactly $6 = n - 1$ medians are known to pass through O directly by construction.

In general, if $P_1 P_2 \ldots P_{\lceil \frac{n}{2} \rceil + 1}$ are $\lceil \frac{n}{2} \rceil + 1$ consecutive vertices of an odd n-gon, then the medians through P_1 and $P_{\lceil \frac{n}{2} \rceil + 1}$ are known since the opposite side of P_1 is $P_{\lceil \frac{n}{2} \rceil} P_{\lceil \frac{n}{2} \rceil + 1}$ and the opposite side of $P_{\lceil \frac{n}{2} \rceil + 1}$ is $P_1 P_2$. Therefore the point of inter-

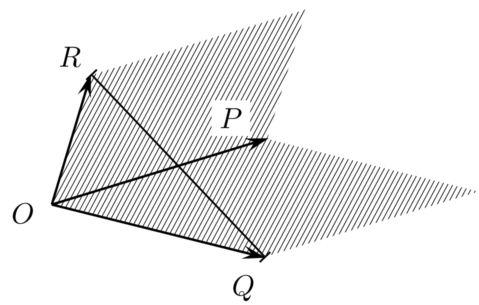

Figure 5

section O of the medians will be known. Now each remaining median of the n-gon will either go through one of the vertices $P_2, \ldots, P_{\lceil \frac{n}{2} \rceil}$ or else through a midpoint of one of the known sides; assuming the medians are concurrent through O, we thus know all their positions. Then by repeated applications of Lemma 3 we may successively determine unique positions for vertices $P_{\lceil \frac{n}{2} \rceil + 2}, \ldots, P_n$ such that side $\overline{P_k P_{k-1}}$ is bisected by $M_{P_{k - \lceil \frac{n}{2} \rceil}}$ for $\lceil \frac{n}{2} \rceil + 2 \leq k \leq n$ (the only remaining median which may not be concurrent through O is thus median $M_{P_{\lfloor \frac{n}{2} \rfloor}}$). Thus we have $n-1$ concurrent medians and Theorem 1 can be applied.

How to prove Theorem 1

Our new shiny proof of Theorem 1 rests on the following simple result:

Lemma 4 *Let O be the origin of the Cartesian plane, and let P, Q and R be three points. Then OP is a median of the segment $\overline{QR}$ if and only if $|PQ| = |RP|$.*

Proof: First assume OP is a median of $\overline{QR}$. Since Q and R are at equal distance from OP, the parallelograms spanned by P and Q and spanned by P and R have equal area (see Fig. 5). So the two determinants $|PQ|$ and $|RP|$ are equal in absolute value, but they must also be the same sign since Q and R are on opposite sides of OP. Therefore $|PQ| = |RP|$.

Similarly if $|PQ| = |RP|$ then Q and R must be at an equal distance from OP and on opposite sides of OP given that the determinants have equal sign. Therefore OP is a median of $\overline{QR}$. ■

We shall now just give a proof of Theorem 1 for the case $n = 5$. Limiting ourselves to $n = 5$ allows us to focus on the ideas rather than on the notation. However, the proof's structure is so transparent that the reader should have no problem generalizing it to other (odd) values of n. The reader may also wish to check that the same proof applies for the case $n = 3$, in which case Theorem 1 simply states that the three medians of a triangle are concurrent.

Proof of Theorem 1 for $n = 5$: Let $ABCDE$ be a pentagon in which all medians—except possibly for M_C—intersect at a point O, which we assume WLOG to be the origin of the Cartesian plane. By Lemma 4 we have that

$$\begin{array}{rcll} |AC| & = & |DA| & \quad \text{(since } OA \text{ is a median)} \\ |DA| & = & |BD| & \quad \text{(since } OD \text{ is a median)} \\ |BD| & = & |EB| & \quad \text{(since } OB \text{ is a median)} \\ |EB| & = & |CE| & \quad \text{(since } OE \text{ is a median)} \end{array} \tag{1}$$

And therefore

$$|CE| = |EB| = |BD| = |DA| = |AC|$$

which implies that OC is a median of $\overline{AE}$ by Lemma 4. ■