Illustrating *Beyond the Third Dimension*

Thomas Banchoff and
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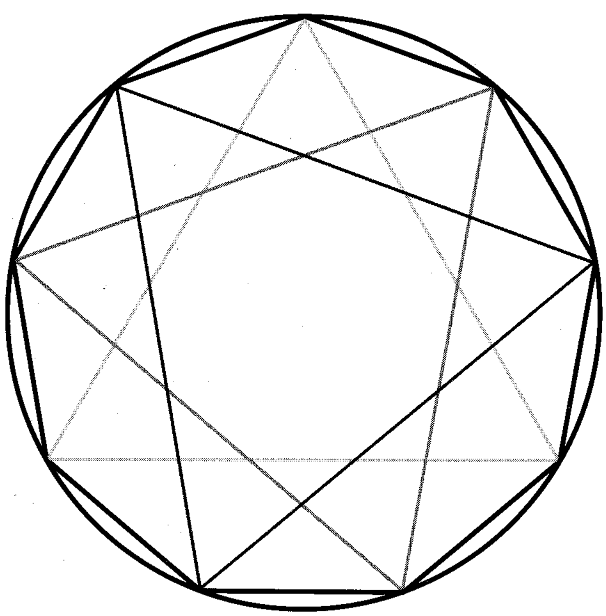
ABSTRACT

Production of the images for the recent volume *Beyond the Third Dimension* involved the use of several new computer-graphics techniques for design of line drawings, and considerable interaction with artists who have worked with higher-dimensional geometric concepts. In this article, the authors describe the tools used and discuss connections with the works of a number of artists.

On 20 June 1990, the first copies of *Beyond the Third Dimension* [1] came off the press. The production of this 210-page volume with 240 illustrations had followed a precise schedule with little variation from the time the preliminary manuscript was sent to the acquisitions editor in August 1989, until the computer disks containing all the finished text and completed line drawings were sent to the printer the following May. In the meantime, nearly every sentence had been rewritten, some of them several times, in response to a thorough editing job by the project editor, which was augmented in later stages by the additional comments of the production editor.

Equal scrutiny was given to the computer-generated artwork, created almost entirely at Brown University, Providence, Rhode Island, United States. In an earlier day, my student assistants and I would have submitted careful drawings to the publisher, who then would have sent them to a design house in New York for preliminary rendering. Then we would have made the corrections (and it is impossible for a design house without a professional geometer on its staff to produce finished drawings of complicated phenomena without introducing numerous mathematical errors). The corrected drawings would then go back to the designers, and this back-and-forth process would be repeated as often as necessary in order to converge to an acceptable solution. In order for this process to be completed in time, the original drawings would have had to be in place before January of 1990. As it was, we contracted the production of computer files of the finished drawings, containing all the information for color separations, so that the final images were designed and implemented as the text was taking its final form in March 1990. This produced a double set of deadlines of which several people at the publishing house predicted we would never be able to meet. Although we seriously underestimated the amount of effort necessary to achieve the goals within the time-frame, we were able to do it, thanks to the hard work of several student assistants and the full cooperation of the production staff at the publishing company.

Fig. 1. The vertices of an enneagon can be separated into three groups, each group forming the vertices of a triangle, indicating a simpler structure underlying the more complicated one.



In this article, we describe some of the more interesting challenges provided by this project, and indicate the methods that we used to produce the original artwork in our volume. We also mention a number of interactions with artists whose work appears in this book, including Attilio Pierelli, Tony Robbin, Max Bill, Lana Posner, David Brisson and Salvador Dalí. Although we will not enter into a full discussion of the mathematical technicalities, we hope to include enough information so that interested persons with access to computer-graphics software can reproduce their own versions of the images described in this article. In our production of the line drawings with shading, we used the personal-computer drawing program Aldus FreeHand [2]. There are several other programs that offer features similar to the ones described below.

Two-Dimensional Geometric Design

Figures from plane geometry provide good exercises for any computer-graphics drawing program. Many such programs can implement familiar construction procedures that go

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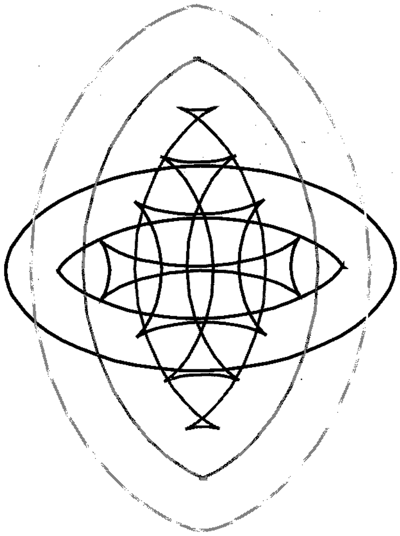


Fig. 2. Wave-fronts emanating from an ellipse. The lighter lines represent waves that are farther from the initial ellipse, which is shown in black.

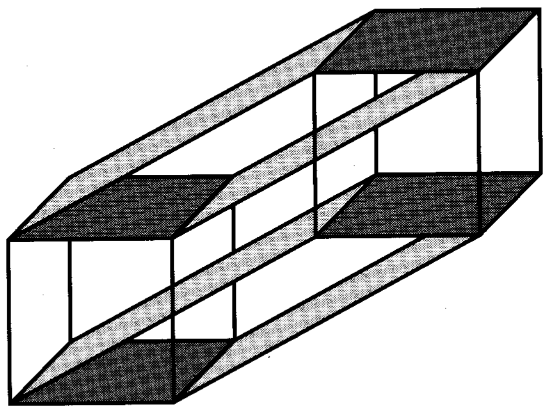


Fig. 3. A representation of a 4D hypercube with two sets of parallel faces highlighted. The figure is easily produced on a computer by duplicating and translating a single initial face in each set.

back to Euclid. We can also utilize computer approximations to produce images that cannot be constructed by the traditional ruler-and-compass procedures, for example, a regular heptagon or a regular enneagon. We can make use of the built-in approximation capabilities of a drawing program to achieve suitable geometric representations. For example, Aldus FreeHand enables us to rotate any figure by any desired angle, so we may rotate by 40° to produce a regular nine-sided polygon, even though it is impossible to construct the 40° angle using ruler and compass alone. That the nine equally spaced points determine three equilateral triangles is a good illustration of an investigation of a complicated figure by finding simpler figures contained within it (Fig. 1).

Preliminary work on the diagrams in our book was done in black and white, a simpler process than that required for color, which we wanted to do since we were also providing illustrations for an extended essay on dimensions for the volume *On the Shoulders of Giants,* published by the National Research Council under the direction of the Mathematical Sciences Education Board [3]. In this book, we had the great advantage of being able to render all diagrams in full color. In principle, our basic colors were restricted to a dozen, chosen in consultation with the members of the publisher's design staff; in fact, we were able to obtain a much larger range of colors by using each of the basic colors in several different concentrations. We used the basic color sequence to encode the steps of several of the more complicated constructions, so that any reader would know that the red portion was drawn first, then the orange, yellow, green, blue and violet portions in sequence. Later we used similar sequencing to describe successive steps in one-parameter families of curves in the plane, when we dealt with wavefronts emanating from a curve such as an ellipse (Fig. 2).

Geometric Design in Dimensions Three and Higher

A number of the diagrams in the book were planar projections of objects in three-dimensional (3D) and four-dimensional (4D) space. We accomplished most of them by traditional drafting procedures, building up orthographic projections by using the machine to transfer segments to parallel positions, so that the squares in the projection of a subdivided cube could be represented by translates of a small number of parallelograms (Fig. 3). Again, the ability of the program to replicate units made it possible to generate complicated objects like exploded views of decompositions of cubes (Fig. 4) and hypercubes, or fold-out versions of 3D polyhedra (Fig. 5) and 4D polytopes (Fig. 6).

Once again, color coding was important for expressing relationships of different parts of a figure or among figures in a family. For example, duals of regular polyhedra were always identified in complementary colors, and we carried the same pattern over as we investigated the regular polytopes in four dimensions. Using various saturations of the basic colors, we could simulate the appearance of shading without having to go through the calculation of the precise

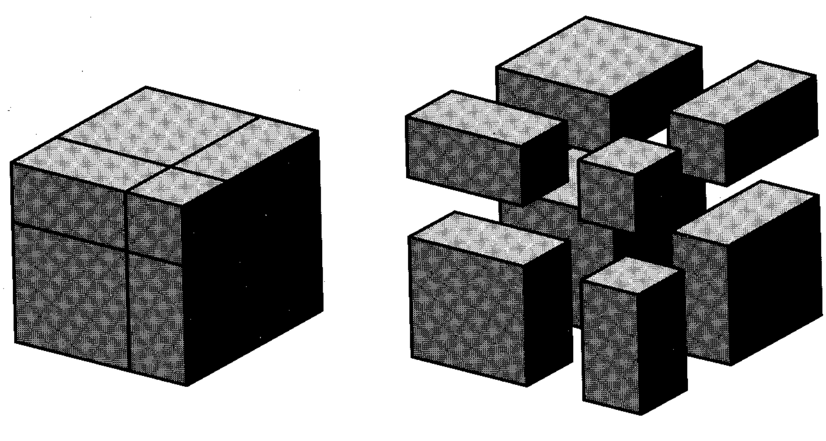


Fig. 4. An exploded view of a cube illustrating the formula for the cube of a binomial. The computer allows each portion of the cube to be treated as a unit, easily allowing its position to be adjusted relative to the other pieces.

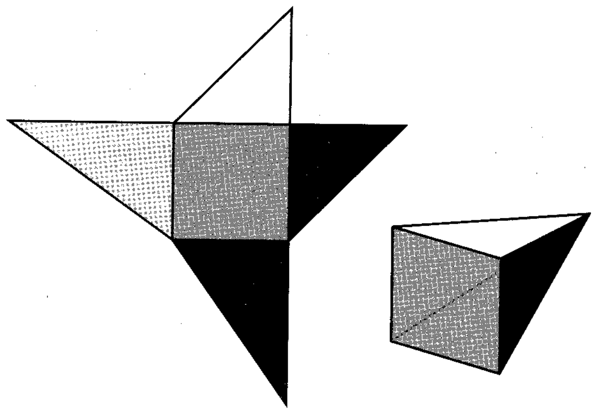


Fig. 5. A planar template that can be folded into a 3D pyramid, which is one-third of a cube.

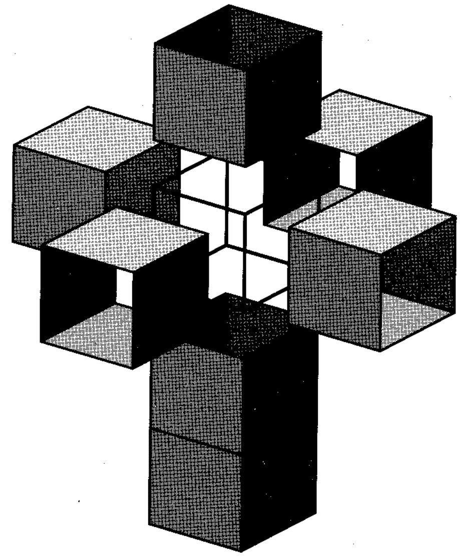


Fig. 6. Just as a 3D cube can be unfolded into six squares that form a cross-shape in the plane, so can the 4D hypercube be unfolded into eight cubes in three-space.

angle of inclination of each face of a polyhedron with the direction of a light source, especially since it is not clear how that shading should be represented in dimensions higher than three.

In only one place in the book did we use anything other than a solid color for each facet of a polyhedron; this was done in the section in which we discuss intersection of two-dimensional (2D) planes in four-space. If we color each point of a surface by its fourth coordinate, then a plane in three-space will appear in monochrome, say violet, while a plane extending into four-space will have a gradation of color expressed as a family of parallel lines going from red to violet to blue. If a two-plane in four-space intersects a plane in three-space at a single point, then the projection of this configuration into three-space will be a pair of intersecting planes, and exactly one point on the violet line in the second plane will intersect the violet plane. This is one of the best ways for identification of the intersection points of a pair of surfaces in four-space by looking at the intersection curves of their projections into three-space (Fig. 7).

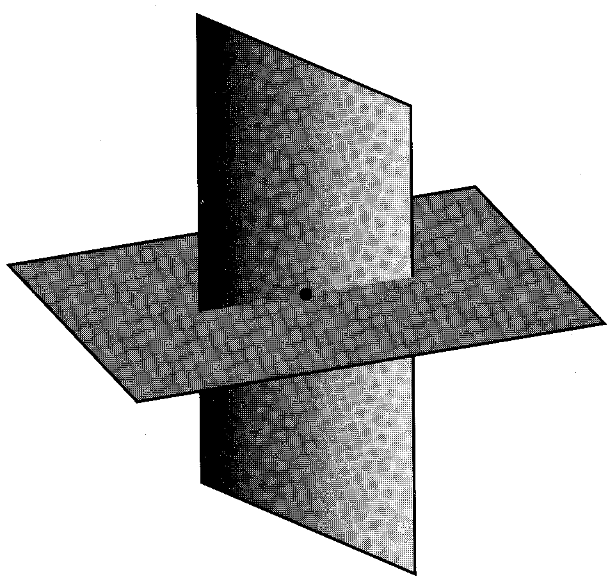
A number of the figures in this book were produced through the use of other computer-graphics devices. Some of the most effective ones resulted from a project, carried out over several years with a student assistant, Nicholas Thompson, who programmed the PRIME PXCL5500 to produce fully rendered images of surfaces in 3D and 4D space. The primary advantage of this technology is its speed of display, permitting real-time rotations, slicing and one-parameter deformations of surfaces as the operator enters various parameters by means of analogue devices such as dials or slider bars. In our book, it was impossible to achieve these animation effects, but we did present storyboards to indicate various scenarios, such as perspective distortions of the images of surfaces rotating in four-space, which we used to illustrate a well-known sequence from the film *The Hypersphere: Foliation and Projections* [4,5].

We were able to use the full-color capabilities of this raster-graphics machine to present other aspects of the geometry of higher dimensions as well. For example, the graph of a complex-valued function of a complex variable requires four real dimensions. In this approach, the graph of the square-root relation is represented as a surface that intersects itself along a segment ending at a pinch point. By coloring the points of the original disc domain according to their angular coordinates, we may easily distinguish the two different sheets passing through a given intersection curve and even identify the coordinates of the angles, for example, when a blue region intersects the orange region halfway around the disc.

Creating the Artwork for *Beyond the Third Dimension*

Two major processes were used to produce the line-art in *Beyond the Third Dimension*. One was to describe the shapes to the computer using mathematical formulas, and have the computer perform 3D-to-2D transformations in order to render a perspective drawing of the objects in question. The other was to use the computer as an electronic canvas with which the operator places each line or shaded region individually, with the precision and flexibility inherent in the computer. The former process might well be called computer-generated artwork, whereas the latter is more properly denoted computer-assisted art. Both processes were critical

Fig. 7. Two planes in four-space may intersect at a single point. Here, the two planes are shown in three-space with the fourth dimension represented by shading. Although the two planes appear to intersect along a line, the only intersection that occurs in four dimensions is at the point where the colors also match.



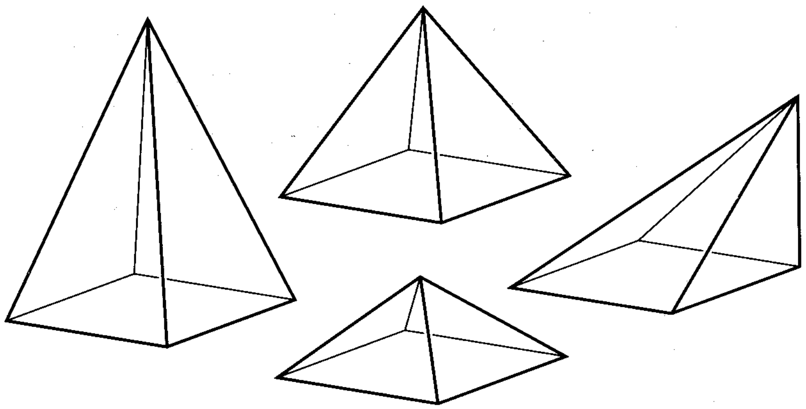


Fig. 8. When the apex of a pyramid is moved, it can be made taller, shorter or skewed. As the vertex is moved, all the lines joining it to the base are automatically redrawn by the computer.

to the production of the book, and each has its advantages and disadvantages.

The computer-generated diagrams appear primarily in chapter 3, simulating the water levels that would occur as different objects sink into a pool. The advantage here is that the computer can generate hundreds of lines and patches far more quickly and accurately than an artist could, and once drawn the picture can be rotated in three dimensions in order to get the best view (the importance of this ability should not be underestimated). There are a number of disadvantages, however. For best results, the computer needs to produce a large number of lines and patches; this requires that the shape of the object be given as a formula, and this may not be easy to do. Second, the artist usually has only global control over the object and cannot manipulate the lines or patches individually. In order to overcome this problem, we decided to combine the computer-generated approach with the computer-aided method—that is, we would use the computer to produce a draft image and then use the illustration program to add the final touches required for the finished piece.

This process introduced a whole set of problems of its own. To begin with, the program that generated the images did not run on a Macintosh personal computer, which we used for producing the text and other artwork, but rather on a Sun workstation. Furthermore, the image-generating program did not produce data files that could be used with our illustration program. Finally, the Macintosh we were using was not attached to the same network as the Sun workstation; this alone made getting the data files to the Macintosh difficult.

Fortunately, the program used to produce the computer-generated artwork was written here at Brown University, and the developers were able to modify it sufficiently to produce data files that could be used on the Macintosh. Only a limited amount of the visual information normally available in the program was transferable, however. The ability to shade the faces was not transportable, for example, which is why all the patches in the diagrams appearing in our book are of a single color. (We have since been able to remove this limitation, but not in time to get the results into the book.)

Once the files were produced, getting them to the Macintosh was not so much difficult as it was tedious. This is much easier now that we have the Macintosh properly networked, but the images for the book had to be moved by computer disk, and, since they were quite large files (on the order of 100K for a single diagram of a torus), this took some time. One reason for such large files is that each line segment and each patch is a separate object within a file, resulting in thousands of individual items in each file. Furthermore, the files included every line and patch, even if they were completely hidden by other ones.

Despite these problems, however, the results were quite good. This combination of computer-generated and computer-aided artwork is very powerful—and is not fully utilized in the computing community. Further software development in this area could be very rewarding.

The remainder of the artwork was perfectly suited to the computer-aided approach. The difficulties here are the same as those faced by an artist using more traditional tools: finding the best view before the object is drawn, getting the perspective right, making sure the sides of a cube are really square, and so forth. Computer tools can be a big help in solving these problems.

Aldus FreeHand is an *object-oriented* drawing program, which means that when a line or a square is drawn on the page, it does not merely become a part of the picture, but retains its identity as a line or square. This means, for example, that the endpoint of the line can be moved, and the entire line will move to the new position. If a square is placed so that it obscures part of a line, the square can be moved to another part of the picture, and the line that was underneath the square will automatically be redrawn by the program. This sharply contrasts with traditional art tools, with which a line drawn on a piece of paper is only so much graphite on the page—if it is necessary to move it, the line must be erased and drawn again somewhere else, possibly destroying lines that intersect it.

Not only can the computer artist move the endpoint of a single line, the endpoints of many lines can be moved at once. For example, in a typical two-point perspective drawing of a cube, all four corners of one face can be selected and moved at once, stretching the cube into a rectangular box. Or the vertex of a pyramid can be moved to create a skewed pyramid or a taller pyramid or a shorter one (Fig. 8). For drawings made up of straight lines and regions filled with solid colors, this is an extremely flexible and powerful environment.

Another important feature of computer-aided illustration involves the ease of duplication of existing objects. For example, once the bottom face of a cube is drawn, the top face can be created simply by duplicating the bottom face and moving it vertically. The lower face itself can be built by drawing the front two edges, then duplicating them and moving the duplicates to form two back edges. Finally, the vertical edges can be duplicates of a single vertical edge at the front corner. We chose to use orthogonal projections rather than perspective drawings for most of the diagrams, partly for mathematical reasons and partly for computational reasons: the procedure outlined above for producing an orthogonal projection of a cube is much easier than the one required to produce a picture of a cube in true perspective. The illustration program we used provides a number of crucial features, including the ability to duplicate movements of objects as well as the objects themselves.

These techniques are enhanced by a feature called *snap guides*. These are vertical or horizontal lines that can be placed anywhere within the picture and that act as magnets for objects; whenever an object is moved near enough to a snap guide, the object will 'snap' to the line. This way it is simple to align multiple objects horizontally or vertically. One of the hardest parts of drawing mathematical objects is getting the ends of lines to match up correctly. This is accomplished easily through the use of horizontal and vertical guides that cross at the point where the lines should join.

One of the most important features for our purposes is the program's ability to resize objects—not just by eye but by specific percentages of the original size—and to have the shrinkage occur toward a specific point. For example, it is simple to find the midpoint of a diagonal line in the following way: first duplicate the line, then shrink the duplicate toward one of its endpoints to 50% of its original size; the other endpoint will now be at the midpoint of the original line—no measurements or calculations are required. To divide a line into thirds, simply duplicate it, then shrink the duplicate to 33% its original size, but this time shrink it toward its center (the program provides this option, so we do not have to find its center first); the endpoints of the duplicate now divide the original line into thirds. For a final example, with a drawing of a pyramid, one can show a horizontal slice halfway up the pyramid, by simply duplicating the base and shrinking it by 50%, this time toward the vertex of the pyramid (Fig. 9). The duplicated base will shrink by 50% and move halfway up the pyramid, both at the same time. As this example indicates, the combination of duplication, alignment and resizing makes these software functions even more powerful.

Although we used a 2D drawing program, it is possible to use its features to produce 3D drawings without too much difficulty. The first step involves the choice of a line segment that represents a unit length in each of the three coordinate directions (for example, the front edge of a cube together with the front two edges of its base). This is one of the harder parts of drawing; it can either be done mathematically, by calculating the exact positions of these initial lines, or by estimating them by hand (in this case, it is usually easiest to draw a cube and adjust it until it looks right, then remove all but the edges described above). Once these lines have been produced, any point in three-space can be located by appropriately resizing and translating these line segments.

The process just described still requires calculating the three-dimensional positions of the points in the drawing (something that computers are well-suited to do, but that humans find tiresome). Fortunately for us, the very geometry of the problems we were illustrating provided much easier methods. For example, in chapter 5 we describe the duals to the regular polyhedra, and the diagrams were constructed in essentially the same way outlined in the text itself. Take, for instance, the cube and its dual, the octahedron. The cube is easily constructed from the three unit-coordinate line-segments, as indicated above; once we have the cube, it is easy to find the centers of its square faces by drawing a diagonal and sizing it to 50% toward one of its endpoints. It is then a simple matter to join these centers together to form the octahedron. The whole process should take only a few minutes for someone practiced in the art of computer-aided drawing, and the result is a beautiful and perfectly accurate mathematical drawing (Fig. 10).

For diagrams with filled-in patches, shading can help make an object look more three-dimensional, however, for objects only made of lines, this effect is harder to achieve. When two lines cross on the page but do not really cross in three-space, visual ambiguity can result. Artists have traditionally solved this problem by breaking the line that is farthest behind, leaving a slight gap around the top line. This makes it clear which line is on top. This effect is achieved on the computer in a surprisingly simple way. Unlike the traditional pencil, the computer can draw in white as well as black; one way to produce the desired broken line is to draw the back line first. Next, draw a wide white line where the top line is to cross the back line, then duplicate it on top of that as a thinner black line. The result will be a black line with a thin white border around it on top of another black line; the white border will obscure the black

Fig. 9. A pyramid can be cut in half by duplicating its base and shrinking it by 50% toward its apex.

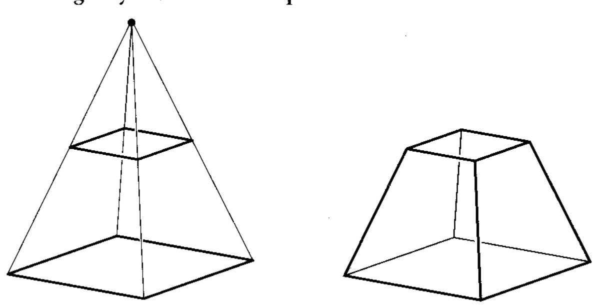
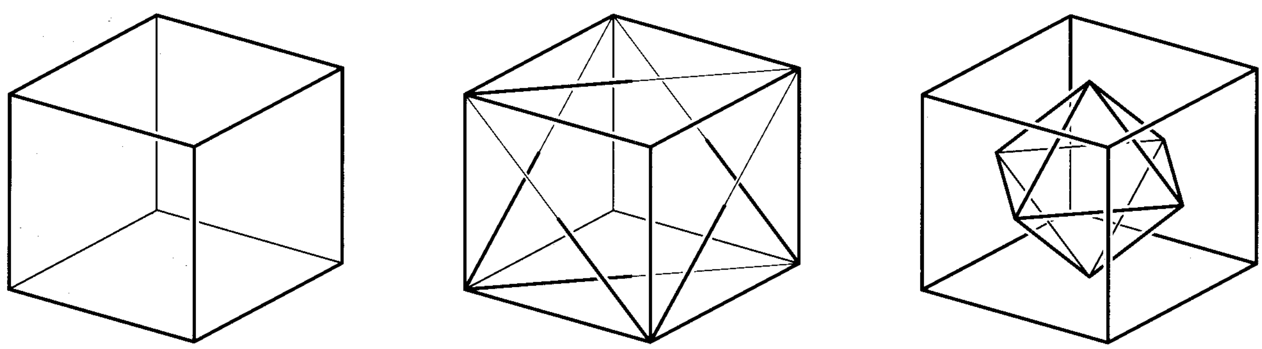


Fig. 10. Beginning with a cube, the centers of each face are found by shrinking a diagonal to 50%, and then these centers are joined to form an octahedron, dual to the cube.



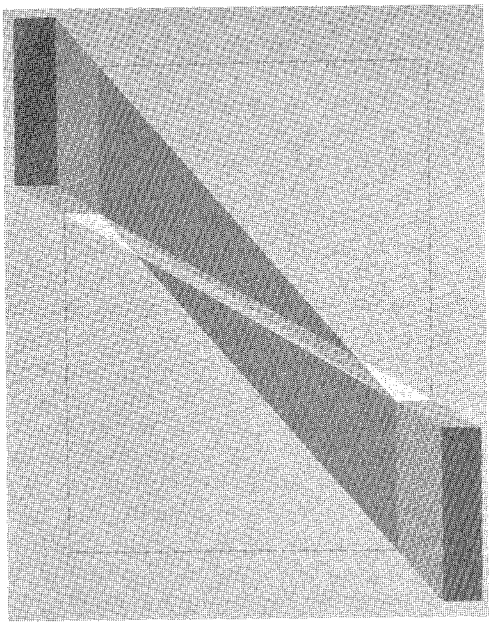


Fig. 11. Lana Posner, *Perspective Twist IV*, acrylic, 24 × 30 in, 1983. This painting shows true perspective in each portion of the figure, but an impossible configuration overall.

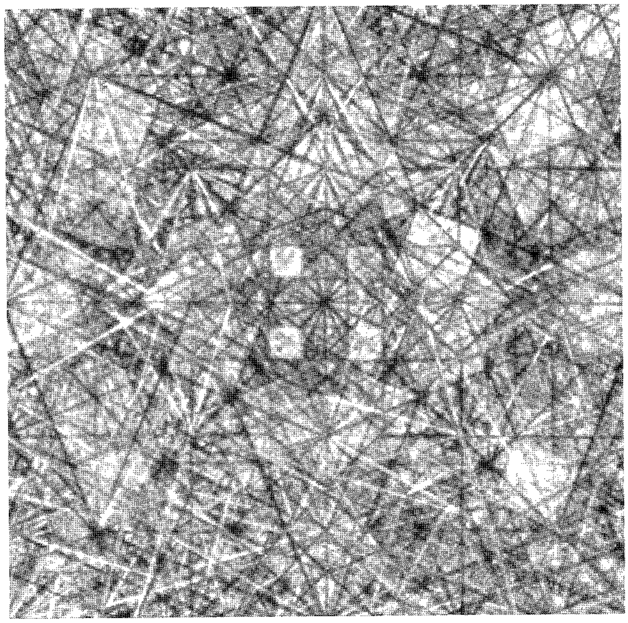


Fig. 12. James Billmyer, Untitled, oil, 30 × 30 in, 1970. Billmyer's linear paintings lead the viewer off the page and back again in four different directions.

line below, resulting in the desired effect of a break. There are a number of advantages to this approach: first, the ends of the broken line will be parallel to the line that crosses it. Second, if the upper line needs to be repositioned for any reason, it is simple to move both the black and the white lines together; thus the 'break' will move along with the line. This makes it convenient to update the picture. On the other hand, some care must be taken near the endpoints of the white line so as not to obscure lines that are supposed to meet at a corner.

This brings us to the final phase of production: corrections and updates. One of the real advantages we enjoyed with the use of computers for artwork production was the ease with which updates were accomplished. If the publisher requested a change to a diagram, it could usually be made the same day. Moreover, it is simple to produce more than one copy of a picture and modify each in slightly different ways in order to consider a number of possible modification options. Each is an original (no paste-ups or overlays), and the best one can be chosen purely on its visual merits. Our publishers were skeptical about our ability to produce the artwork in-house, and the rate at which we were able to produce corrections amazed them.

The original diagrams were drawn at whatever sizes were most convenient for the artist. The publisher used photo-reductions to lay out the book, and then gave us the measurements for the final diagrams. The ability of the illustration program to resize objects (in this case, the entire diagram) came in handy again, as it allowed us to reduce the original drawings to exactly the required sizes. Photo-reducing would have reduced not only the overall size of the diagram, but also the width of each line and the size of the text in the picture. The illustration program, however, allowed us to reduce the size of the diagram while leaving the line widths unchanged; thus line-sizes were consistently maintained from figure to figure.

THE ARTISTS OF *BEYOND THE THIRD DIMENSION*

Many artists have permitted their work to be displayed in the book as illustrations of the interaction between dimensionality and art. Lana Posner and I began working together 10 years ago when her work was displayed at the Providence Art Club. The use of impossible figures to explore perception of form is illustrated most clearly in her painting *Perspective Twist* (Fig. 11). Although each portion of the painting suggests a natural three-dimensional interpretation, there is no consistent interpretation of the configuration as a whole, a phenomenon explored most completely and effectively by

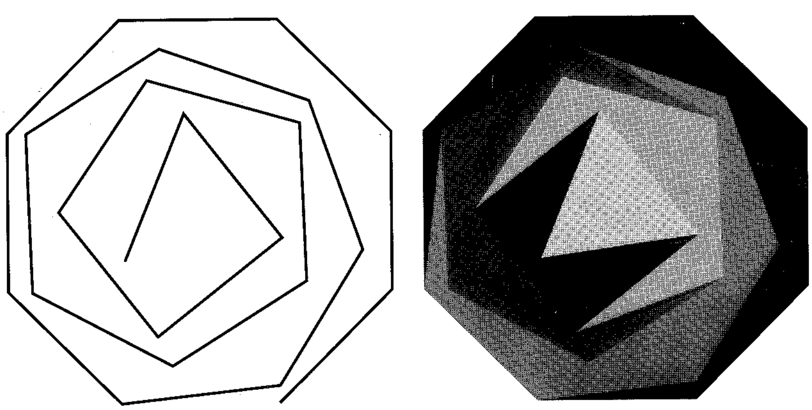


Fig. 13. Computer graphic based on Max Bill's forms depicted in his lithographs *The Theme* and *Variation 1* of 1938. Equal-sided polygons propose outward from a triangle in the center to a regular octagon.

M. C. Escher [6–9]. Using the computer, we investigated the challenge of realizing this image as the projection of a figure in space by arbitrarily assigning additional coordinates to vertices of the image, either preserving central symmetry or assigning the same additional coordinates to points situated symmetrically with respect to the center of the picture. The assigning of one additional coordinate to each point lifted the picture into three-space, necessarily introducing self-intersections, since no consistent lifting was possible. The assigning of two additional coordinates, however, made it possible for us to see the original image as the projection into the plane of a figure in four-space that had no self-intersections whatsoever. Rotation of this figure through different complete turns provided families of images, coalescing back to the original. We did not work long enough to determine which choices of additional coordinates would provide the most artistically pleasing four-dimensional sculpture based on the original image, but this does seem to be a good project.

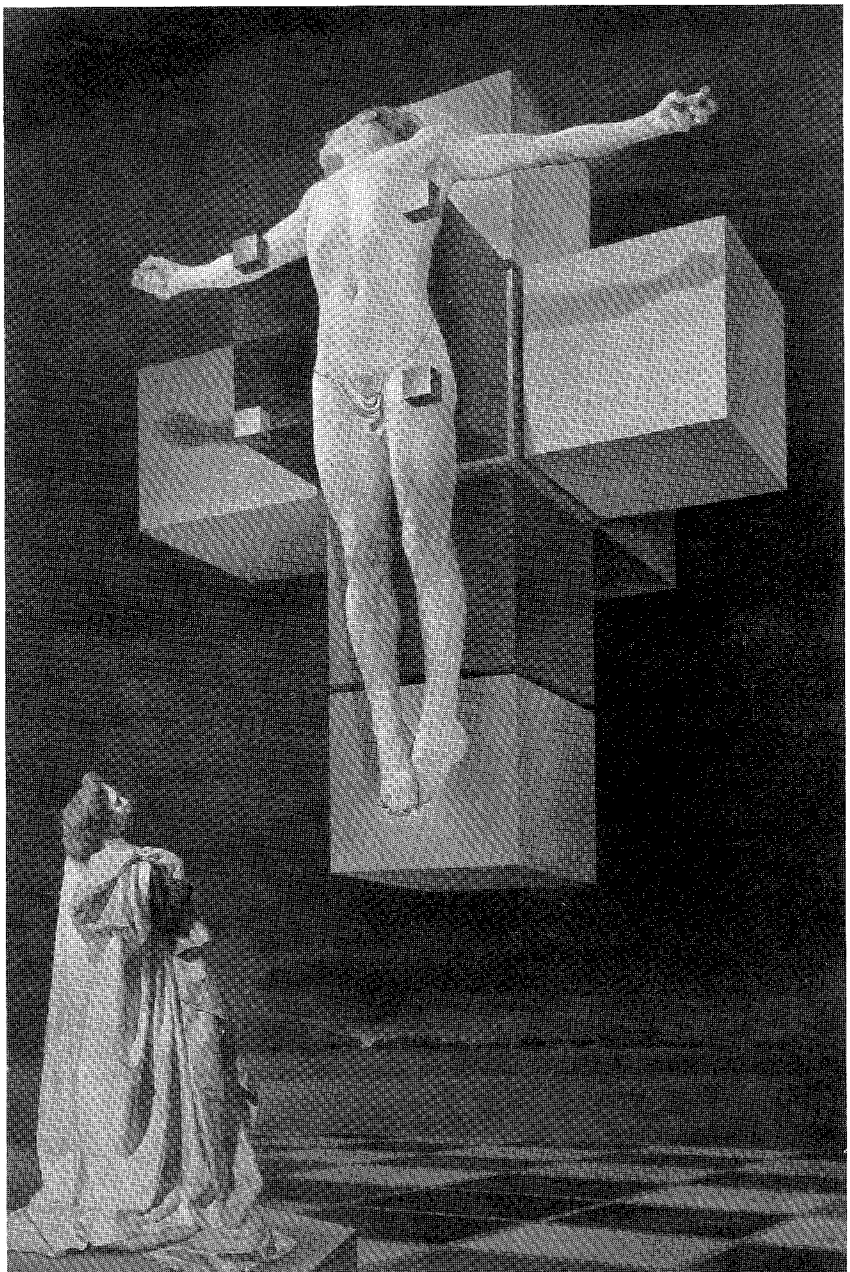
Paintings that come off the canvas in different ways appear in other guises as well. The work of the late James Billmyer is a particularly impressive illustration of the way a two-dimensional oil painting can take on higher-dimensional significance. Billmyer worked for a number of years with Hans Hofmann, an artist who stressed the importance of having objects move out from the canvas and then resolve back into it. By dealing with multiple rhythms, each with its family of angles on the plane and each with its representative color, Billmyer created patterns that would take the viewer out of the plane in independent directions before resolving back to the plane (Fig. 12). Billmyer and I spent many fascinating hours examining the higher-dimensional patterns that arose from the canvas, reminiscent of the forms that appeared in the film *The Hypercube: Projections and Slicing* [10].

Max Bill carried out a series of investigations of polygonal forms in the 1930s, and one image in particular seems to illustrate the increasing sequence of n-gons, spiralling out from a generating triangle (Fig. 13) [11]. According to Bill, the original painting was inaccurately reproduced when it was first published in Paris in 1938. With his permission, my assistant and I created a version of his image with the colors in the palette selected for our volume.

We had originally intended to use a photograph of a Bill sculpture of a Möbius band as the opening image for the final chapter on non-Euclidean geometry and nonorientable surfaces [12–14]. Unfortunately the photolithograph of the artist's work that we purchased from the publisher turned out to be too large for our page, so it could not be used. Fortunately there was a good replacement, the Klein bottle rendered in glass by William D. Clark, a retired physicist living in California. Over the years there have been any number of glass-blown realizations of this important surface, but none has been more effective than the one developed over many years by Clark. The photograph of his image with a blue background is especially effective for displaying its beautifully constructed form.

Also appearing in an extremely attractive photograph is the stainless steel sculpture, *Ipercubo,* (the hypercube) by Attilio Pierelli, a sculptor residing in Rome (Color Plate C No. 3). As the leader of the *Dimensionalismo* art movement in Italy, he remains at the forefront in rendering the central projections of four-dimensional figures in materials that bring out their structure in new ways, with multifaceted reflections that reveal new symmetries as the object turns or as the viewer walks around it. Some of these effects notably appear in the film *Dimensions* [15].

Fig. 14. Salvador Dalí, *Corpus Hypercubicus* (The Crucifixion), oil on canvas, $76\frac{1}{2} \times 48\frac{3}{4}$ in, 1955. (Metropolitan Museum of Art, gift of Chester Dale Collection)



Tony Robbin's work is beautifully reproduced in a photograph in this volume, but no single picture can begin to do justice to the experience of viewing his structures from different angles [16]. As the viewer moves, the shadows of 3D wire figures emanating from the canvas play against the 2D acrylic painting. Other works of Robbin explore this phenomenon of the interaction of sculpture and painting more fully by allowing the viewer to see red or green shadows with anaglyph glasses, providing two different projections.

This concept of providing different images to the left and right eyes is, of course, the basis of ordinary stereoscopic viewing. Extraordinary, however, is the description of hyperstereoscopic vision, as developed by the late David Brisson, founder of the Hypergraphics Group, which has brought together the efforts of many painters, sculptors and filmmakers, all inspired by the challenge of representing geometry of one dimension in the medium of another [17] (Color Plate C No. 4). A single hyperstereogram represents the projection of a figure from four-space into a pair of planes that do not meet in a line parallel to the spine of the viewer, as in ordinary stereoptic viewing, but rather in a single point. In viewing such a hyperstereogram, the observer sees different parts of the object in focus depending on the sideways inclination. The insights to be gained by a systematic and complete investigation of this remarkable phenomenon are only beginning to be appreciated. The film *Dimensions* mentioned above is dedicated to Brisson, who appears in the film along with his wife, sculptor Harriet Brisson [18].

My most fascinating connection with artists interested in dimension is my association with Salvador Dalí, beginning in 1976 and extending over a series of a dozen visits until 1986, 2 years before his death. For a number of years, my computer-graphics colleague Charles Strauss and I had displayed Dalí's *Corpus Hypercubicus* (Fig. 14) as an example of the way artists use higher-dimensional imagery in conscious ways in their paintings. When the *Washington Post* ran a story on our work, they included in the background a photograph of that surrealist painting. Shortly after that, we received a call inviting us to New York to confer with Dalí, who was interested in ways of creating and presenting stereoscopic oil paintings. We were impressed by the level of his technical knowledge, and he was impressed in particular by the folding model of the hypercube that was devised as a part of my 1964 thesis on global differential geometry. He kept the model, and subsequently included a copy of it in his museum in Figueres, Catalonia, Spain. When he was asked about the inspiration for the *Corpus Hypercubicus,* he referred to the philosophy of Raimondo Lulio [19], coincidentally already familiar to me. That coincidence produced a mutual respect that led to a series of invitations from Dalí over the next few years to show our new slides, videotapes and films, and to see his new works in progress. Strauss and I began at this time to work on the project, described in detail in the book, on the use of perspective to explore illusions, not just in the plane but in three-dimensional space in a variety of scales. By the time we visited Dalí in Paris in 1981, the project had literally gone off the earth, never to be built. Although I discussed this and other projects with Dalí in three further visits to Spain, we never completed a project together. Dalí did always seem to enjoy viewing the films, and even toward the end he still delighted in identifying the elliptic and hyperbolic catastrophes that appeared as light caustics in computer-generated wire-frame images of surfaces projected from four-space.

CONCLUSION

Working on the production of images for *Beyond the Third Dimension* has given us an opportunity to develop a whole range of techniques for dealing with the geometry of different dimensions and, at the same time, establish new relationships with a variety of artists. We look forward to more experiences of the same sort in the future.

References and Notes

1. Thomas Banchoff, *Beyond the Third Dimension* (New York: Freeman, 1990). Article co-author Davide P. Cervone was in charge of image production for *Beyond the Third Dimension.*

2. Aldus FreeHand, published by Aldus Corporation for the Macintosh computer, is a powerful and full-featured two-dimensional drawing program.

3. Lynn Steen, ed., *On the Shoulders of Giants* (Washington, D.C.: Mathematical Sciences Education Board, National Academy Press, 1990).

4. T. Banchoff, H. Koçak, D. Laidlaw and D. Margolis, *The Hypersphere: Foliation and Projections,* 16mm color-silent film, 3 min (Providence, RI: Brown University, 1985).

5. For a description of a sequence from *The Hypersphere: Foliation and Projections,* see Michele Emmer, *Lo spazio tra matematica ed arte,* exh. cat. of the *Spazio* (space) section of La Biennale d'Arte di Venezia (Venice: Edizioni La Bienalle, 1986) pp. 37–39.

6. L. S. Penrose and R. Penrose, "Impossible Objects: A Special Type of Visual Illusion", *Brit. J. Psych* **49** (1958) pp. 31–33.

7. R. Penrose, "Escher and the Visual Representation of Mathematical Ideas", in H. S. M. Coxeter, M. Emmer, R. Penrose and M. L. Teubner, eds., *M. C. Escher: Art and Science* (Amsterdam: North Holland, 1988).

8. M. Emmer with R. Penrose, *M. C. Escher: Geometries and Impossible Worlds,* 16mm color-sound film from the series *Art and Mathematics,* 27 min (Rome: FILM 7 International, 1984).

9. M. Emmer, "*The Belvedere* by Escher: a Modest Hypothesis", *Structural Topology* **17** (1991).

10. T. Banchoff and C. Strauss, *The Hypercube: Projections and Slicing,* 16mm color-sound film, 9 min (Providence, RI: Brown University, 1978).

11. Max Bill, *Fifteen Variations on a Single Theme,* 16 lithographs, 30.5 × 32 cm, 1934–1938. Originally published as *Quinze variations sur un même théme* (Paris: Editions des Chroniques du Jour, 1938). Reprinted in E. Huttinger, *Max Bill* (Zurich: ABC Editions, 1978). See the front and back covers of this Special Issue of *Leonardo,* for *Variation 10* and *The Theme;* and p. iv for a description of Bill's series of lithographs.

12. M. Emmer with Max Bill, *Ars Combinatoria,* 16mm color-sound film from the series *Art and Mathematics,* 27 min (Rome: FILM 7 International, 1982).

13. M. Emmer with Max Bill, *Möbius Band,* 16mm color-sound film from the series *Art and Mathematics,* 27 min (Rome: FILM 7 International, 1982).

14. M. Emmer, "Visual Art and Mathematics: The Möbius Band", *Leonardo* **13**, No. 2, 108–111 (1980).

15. M. Emmer with Thomas Banchoff, *Dimensions,* 16mm color-sound film from the series *Art and Mathematics;* 27 min (Rome: FILM 7 International, 1984).

16. For a discussion of the four-dimensional elements in his paintings, see Tony Robbin, "Painting and Physics: Modeling Artistic and Scientific Experience in Four Spatial Dimensions", *Leonardo* **17**, No. 4, 227–233 (1984).

17. D. Brisson, ed., *Hypergraphics: Visualizing Complex Relationships in Art, Science and Technology* (Washington, D.C.: American Association for the Advancement of Science, No. 24, 1978).

18. See Harriet Brisson's article, "Visualization in Art and Science" in this issue of *Leonardo.*

19. For the work of the twelfth-century Catalonian, Raimondo Lulio, see Juan de Herrera, *Discurso de la Figura Cubica* (Madrid: Editorial Plutarco, 1935).