Tutorial: Notes on Nonlinear Stability

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1 Introduction

Consider a nonlinear evolution equation

$$\frac{du}{dt} = A(u)$$

and an equilibrium solution ϕ; that is, $0 = A(\phi)$. A concept of central importance in many branches of science is the concept of stability.

Definition. The equilibrium ϕ is **stable** (that is, nonlinearly stable) if: $\forall \epsilon > 0, \exists \delta > 0$ such that if $\|u_0 - \phi\|_1 < \delta$, then there exists a unique solution $u(t)$ with $u(0) = u_0$ defined for $0 \le t < \infty$ such that

$$\sup_{0 \le t < \infty} \|u(t) - \phi\|_2 < \epsilon.$$

It is **unstable** if it is not stable.

The definition may be very sensitive to the norms $\|...\|_1$ and $\|...\|_2$ as well as to the space in which $u(\cdot)$ exists! If X is a Banach space, we define stability in X to mean that X is chosen in all three places. The definition must be modified for non-equilibrium solutions such as traveling waves (orbital stability). The definition must be modified in case some solutions do not exist for all time.

In case an orbit is unstable, a deep question is the following. What happens to it as $t \to +\infty$? Does it blow up? Does it converge to another equilibrium?

Linearization.

Definition. Consider the linear equation

$$\frac{dv}{dt} = Lv \qquad \text{where} \quad L = A'(\phi).$$

The equilibrium ϕ is **linearly stable** if: $\forall \epsilon > 0, \exists \delta > 0$ such that

$$\text{if } \|v(0)\|_1 < \delta, \text{ then } \sup_{0 \le t < \infty} \|v(t)\|_2 < \epsilon.$$

Again the definition depends on the norms! Why is linearization relevant to nonlinear stability? We introduce the notation $w = u - \phi$ for the difference between a solution and the equilibrium. In terms of w, equation (1) can be written as

$$\frac{dw}{dt} = Lw + F(w)$$

where $L = A'(\phi)$ and $F(w) = O(|w|^2)$ formally. The idea is that so long as $w(t)$ remains very small, the nonlinear part $F(w)$ is negligible. The mathematical problem is to investigate whether or not $w(t)$ remains small. Our basic theme is the question: Can we prove nonlinear (in)stability directly? Or does linear (in)stability imply nonlinear (in)stability? In what norms?

Example. Consider the PDE $u_t = xu_x + u^2$ for $x \in \mathbb{R}$ and its equilibrium solution $\phi = 0$. Consider solutions that vanish in some manner as $|x| \to \infty$. Its linearized equation $v_t = xv_x$ satisfies $\int v^2 dx = ce^{-t}$. This comes from multiplying by v and integrating. Hence it is linearly stable in the L^2 norm. Nevertheless the solutions of the nonlinear PDE blow up (in particular, at $x = 0$). This example shows how carefully the norms have to be chosen!

1.1 Finite dimensions

All norms are equivalent.

Theorem 1.1. *Let $A(\phi) = 0$. Assume that A is of class C^1. If all the eigenvalues of L are in the open left half-plane $Re\lambda < 0$, then ϕ is nonlinearly stable.*

Proof. Say $X = \mathbb{R}^m$. Let $\lambda_1, \lambda_2, \ldots$ be the eigenvalues of L. There might be multiplicities. Let $max_j \lambda_j < -\alpha < 0$. Then there is a constant C such that the exponential matrix satisfies

$$\|e^{Lt}\| \le Ce^{-\alpha t} \quad \forall t \ge 0.$$

Then we have $d(u-\phi)/dt = L(u-\phi) + F(u) - F(\phi)$, so that

$$u(t) - \phi = e^{Lt}[u(0) - \phi] + \int_0^t e^{L(t-s)}[F(u(s)) - F(\phi)]ds$$

and

$$|u(t) - \phi| \le Ce^{-\alpha t}|u(0) - \phi| + C\int_0^t e^{-\alpha(t-s)}|F(u(s)) - F(\phi)|ds.$$

Let $|u(0) - \phi| < \delta$ and

$$T = \sup\{r \ : \ u \in C([0,r]; X), \quad |u(t) - \phi| < 2C\delta e^{-\alpha t}, \ \forall t \in [0,r]\}.$$

If δ is sufficiently small, then

$$|F(u(s)) - F(\phi)| \le C_1|u(s) - \phi|^2 \le C_1\{2C\delta e^{-\alpha s}\}^2 \quad \forall s \in [0,T).$$

Thus for $0 \le t < T$ we have

$$|u(t) - \phi| \le Ce^{-\alpha t}\delta + 4C_1C^3\delta^2 e^{-\alpha t}\int_0^t e^{-\alpha s}ds \le \{C\delta + 4C_1C^3\alpha^{-1}\delta^2\}e^{-\alpha t}.$$

Choosing δ sufficiently small, the last expression is less than $2C\delta e^{-\alpha t}$. Given any $\epsilon > 0$, we also choose $\delta < \epsilon(2C)^{-1}$ so that the last expression is less than ϵ. Thus $T = \infty$ and $|u(t) - \phi| \to 0$ as $t \to \infty$. □

Theorem 1.2. *If there is an eigenvalue of L in the open right half plane, $Re\lambda > 0$, then ϕ is nonlinearly unstable.*

Proof. Let $Lv = \lambda v$, $\Re\lambda > 0$. Let λ have the maximal real part. In general, λ and v are complex: $v \in \mathbb{C}^m$. We consider complex-valued solutions. For simplicity take $\phi = 0$. Let $u(0) = \delta v$ for some small $\delta > 0$. Write $du/dt = Lu + (A(u) - Lu)$ or

$$\|u(t) - e^{Lt}\delta v\| \le \int_0^t \|e^{L(t-s)}\| \ \|A(u(s)) - Lu(s)\|ds \le C_\epsilon \int_0^t e^{(\epsilon + Re\lambda)(t-s)}\|u(s)\|^2 \ ds.$$

so long as $u(t)$ remains close enough to 0. Then show that $e^{Lt}v = e^{\lambda t}v$ dominates the nonlinear term. □

Exercise: Complete the instability proof. Also prove it for $\phi \neq 0$ and for real-valued solutions.

Example: the harmonic oscillator. Take $X = \mathbb{R}^2$ and the equation

$$\frac{du}{dt} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} u$$

Its eigenvalues are $\pm i$. This does not provide enough information: nonlinear perturbations can be either stable or unstable!

In this tutorial we will mostly consider dispersive waves, which roughly means that most of the spectrum of L is imaginary.

2 First example of stability: Peakons

We shall see that a fundamental tool for many stability problems is a Liapunov functional, that is, a functional which is monotonically non-increasing (in t) along all solutions. A special case is an invariant functional (conservation law). (Liapunov functionals typically occur in parabolic PDEs, which have spectra in the left half plane. Invariant functionals typically occur in hyperbolic or dispersive PDEs and are associated with spectra at the origin.)

Here is a weird-looking equation but it has a very simple direct proof of stability using two invariant functionals.

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{2.1}$$

"Camassa-Holm equation". Look for a traveling wave solution $u(x,t) = \phi(x - ct)$, $c =$ constant, decaying at ∞. Get an ODE. Integrate twice to get $\phi_x^2 = \phi^2$. Thus we get the traveling wave solutions ("peakons")

$$u = ce^{-|x-ct|}, \qquad c > 0.$$

One should legitimately object to taking 2nd and 3rd derivatives, so let's rewrite the PDE as

$$u_t + \frac{1}{2}\partial(1-\partial^2)^{-1}\{3u^2 - 2(uu_x)_x + u_x^2\} = 0$$

where $\partial = \partial/\partial x$ and where $\partial(1-\partial^2)^{-1}$ is an explicit integral operator. There are many interesting properties of this equation, such as its complete integrability. Another property is that for many of its solutions, u_x blows up in a finite time $T = T(u)$. However, we will concentrate only on the question of stability.

Theorem 2.1. *The solitary waves are nonlinearly stable in $H^1(\mathbb{R})$ in the following sense: $\forall \epsilon > 0, \exists \delta > 0$ such that if $u \in C([0,T); H^1(\mathbb{R}))$ is a solution of the PDE with $\|u(0) - \phi\|_{H^1(\mathbb{R})} < \delta$, then*

$$\sup_{0 \le t < T} \inf_{\xi \in \mathbb{R}} \|u(t) - \phi(\cdot - \xi)\|_{H^1(\mathbb{R})} < \epsilon.$$

Lemma 2.2. *There are two invariants (independent of time for any solution)*

$$E(u) = \int_{\mathbb{R}} (u^2 + u_x^2)dx, \quad F(u) = \int_{\mathbb{R}} (u^3 + uu_x^2)dx.$$

We leave the proof as an **exercise**. The calculation is easier if you first verify that the PDE can be written as $u_t + JF'(u) = 0$ where $J = \frac{1}{2}\partial(1-\partial^2)^{-1}$.

In the sequel we may as well take $c = 1$. Let $\phi(x) = e^{-|x|}$ so that $\phi_x^2 = \phi^2$ and $\phi - \phi_{xx} = 2\delta$.

Lemma 2.3. *For all $u \in H^1(\mathbb{R})$ and $\xi \in \mathbb{R}$,*

$$E(u - \phi(\cdot - \xi)) = E(u) - E(\phi) + 4(u(\xi) - 1).$$

Proof.

$$\begin{aligned} E(u-\phi) &= E(u) + E(\phi) - 2\int (u_x\phi_x + u\phi)dx \\ &= E(u) + E(\phi) - 2\int u(-\phi_{xx} + \phi)dx \\ &= E(u) - E(\phi) - 4(u(0) - 1) \end{aligned}$$

since $-\phi_{xx} + \phi = 2\delta$ and $E(\phi) = 2$ and where we have taken $\xi = 0$. □

Lemma 2.4. *Fix $u \in H^1(\mathbb{R})$ and let $M(u) = \max_x u(x)$. Then*

$$F \leq ME - \frac{2}{3}M^3.$$

Proof. Note that $F \leq ME$ is trivial. Let $g = u + \operatorname{sign}(x-\xi)\, u_x$. Then if $u(x)$ is maximized at ξ, we calculate $\int g^2 dx = E(u) - 2M^2(u)$ and $\int ug^2 dx = F(u) - \frac{4}{3}M^3(u)$. Therefore $F - \frac{4}{3}M^3 \leq M\int g^2 dx = ME - 2M^3$. □

It is interesting that the inequality $0 \leq E - 2M^3$ is identical to the Sobolev inequality in 1D. So among all functions of fixed energy E, the peakon is the tallest!

Lemma 2.5. *Fix $u \in H^1(\mathbb{R})$. If $E(u)$ is near $E(\phi) = 2$, then $F(u)$ is near $F(\phi) = 4/3$.*

We leave the proof as an **exercise**.

Lemma 2.6. *Fix $u \in H^1(\mathbb{R})$. If $E(u)$ is 3δ-near 2 and $F(u)$ is 5δ-near $4/3$, then $|M(u)-1| \leq 2\sqrt{\delta}$.*

Proof. We have $M(u)^3 - \frac{3}{2}M(u)E(u) + \frac{3}{2}F(u) \leq 0$. Equality occurs when $u = \phi$, in which case $M = 1, E = 2, F = 4/3$. Look at the graph of the cubic

$$Q(y) = y^3 - \frac{3}{2}Ey + \frac{3}{2}F = (y-1)^2(y+2)$$

for $E = 2, F = 4/3$. It has a local minimum at $y = 1$. If we drop the graph slightly, the value of $M(u)$ must lie between the two roots near the minimum. □

Proof of theorem. Let $\epsilon > 0$ and let $\|u(0) - \phi\|_{H^1} < (\epsilon/3)^4 \equiv \delta$. By Lemma 2.5 applied to $u(0)$, $E(u)$ is near 2 and $F(u)$ is near $4/3$. By Lemma 2.6 applied to $u(t)$, $|M(u(t)) - 1| \leq 2\sqrt{\delta}$. By Lemma 2.3,

$$\|u(t) - \phi(\cdot - \xi(t)\|_{H^1}^2 = E(u(t)) - E(\phi) - 4M(u(t)) + 4 \leq 3\delta + 4(2\sqrt{\delta}) < \epsilon^2. \quad \square$$

3 Euler equation 2D

3.1 Classical theory

We consider the Euler equation, derived by Euler about 1750 for an inviscid incompressible fluid in two dimensions. Let u be the velocity at a point and p be the pressure. The equations are

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0, \quad \nabla \cdot u = 0.$$

They hold in the region $\Omega \subset \mathbb{R}^2$ where the fluid is located. The first equation expresses the conservation of momentum and the second the incompressibility. We assume the domain represents a closed impermeable vessel; thus the boundary condition is $u \cdot n = 0$ where n is the unit outer normal.

Since we have assumed it is two-dimensional, the vorticity is

$$\omega = \nabla \times u = \partial_x u_2 - \partial_y u_1.$$

It satisfies

$$\frac{\partial \omega}{\partial t} + u \cdot \nabla \omega = 0.$$

The stream function ψ is defined (up to a constant) by $u = [\partial_y \psi, -\partial_x \psi]^T$ provided the domain is simply-connected. Thus

$$-\Delta \psi = \omega \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega.$$

So for a steady-state we have both $u_0 \cdot \nabla \psi_0 = 0$ and $u_0 \cdot \nabla \omega_0 = 0$. So ψ_0 and $\omega_0 = -\Delta \psi_0$ have parallel gradients. So

$$-\Delta \psi_0 = \gamma(\psi_0)$$

at least locally.

The basic question we address is: *which steady states u_0 are stable and which are unstable?* That is, if a flow starts out near a steady state u_0, does it remain nearby for all time? This is called "nonlinear stability". Instability may be a precursor to the outset of turbulence.

A very simple steady state in a strip $\{(x, y) : a < y < b\}$ is the (parallel) shear flow $u_0 = [U(y), 0]^T$ for any function $U(y)$.

Theorem 3.1 (Rayleigh's Criterion (1880))**.** *If $U(y)$ has no inflection point, then the shear flow is linearly stable.*

Proof. Linearize the equation for the stream function ψ, and look for exponential solutions

$$\psi = e^{i\alpha(x-ct)} f(y)$$

with $\alpha \in \mathbb{R}, c \in \mathbb{C}$ and $f(y) \in \mathbb{C}$. We want to prove there is no solution $\psi \not\equiv 0$ with $\Im c > 0$. It reduces to the ODE (Rayleigh's equation)

$$(U(y) - c)(f'' - \alpha^2 f) = U'' f$$

with $f(a) = f(b) = 0$. **Exercise**: Derive this equation, making use of the two-dimensionality.

It is almost of Sturm-Liouville type, except that the coefficient can vanish. Multiply the ODE by $(U(y) - c)^{-1}\overline{f(y)}$ and then integrate over y and take the imaginary part to obtain

$$(\Im c) \int_a^b \frac{U''(y)|f(y)|^2}{|U(y) - c|^2} dy = 0.$$

Hence, if $U'' \neq 0$ and U'' is continuous, then $\Im c = 0$. □

Theorem 3.2 (Arnold's Criterion (1965)). *Let $\Omega \subset \mathbb{R}^2$ be a bounded simply-connected open set. Let a steady state $-\Delta\psi_0 = \omega_0 = \gamma(\psi_0) = \nabla \times u_0$ be given with $u_0 \cdot \nu = 0$ on $\partial\Omega$. If γ is single-valued and $\gamma' < 0$, then u_0 is nonlinearly stable in the $H^1(\Omega)$ norm.*

For instance, in the case of a shear flow, $U(y) = \psi_0'(y)$ so that no inflection point of U means that $\gamma'(\psi_0)$ vanishes nowhere. So Arnold's Criterion is close to a nonlinear (and more rigorous) version of Rayleigh's Criterion.

Proof. Use the temporal invariant

$$A(u) = \frac{1}{2} \int_\Omega |u|^2 dx + \int_\Omega H(\omega) dx$$

where H can be anything. Here we choose $h = -\gamma^{-1}$ and $H' = h$. Note that this H is strictly convex. By invariance,

$$A(u(0)) - A(u_0) = \frac{1}{2} \int (|u(t)|^2 - |u_0|^2) dx + \int (H(\omega(t)) - H(\omega_0)) dx.$$

By convexity $\exists c_0 > 0$ (why is it uniform?) such that

$$H(\omega) - H(\omega_0) \geq H'(\omega_0)(\omega - \omega_0) + c_0(\omega - \omega_0)^2.$$

Also we estimate

$$\begin{aligned} \int_\Omega H'(\omega_0)(\omega - \omega_0) dx &= -(\gamma^{-1}(\omega_0), \omega - \omega_0) = (\psi_0, \Delta\psi - \Delta\psi_0) \\ &= -(\nabla\psi, \nabla\psi_0) + (\nabla\psi_0, \nabla\psi_0) = -(u_0, u) + |u_0|^2. \end{aligned}$$

We deduce that

$$\epsilon > A(u(0) - A(u_0) \geq \sup_t \int_\Omega \left\{ \tfrac{1}{2}|u(t) - u_0)|^2 + c_0(\omega(t) - \omega_0)^2 \right\} dx.$$

So $u(t)$ remains near u_0 in the H^1 norm. □ □

Exercise: What happened to the boundary term?

3.2 Linearization can be a tricky business

The incompressible Euler equation is

$$\partial_t u + (u \cdot \nabla)u + \nabla p = 0, \qquad \nabla \cdot u = 0$$

where $x \in \Omega \subset \mathbb{R}^2, u \in \mathbb{R}^2$. Here Ω is a smooth, bounded, simply connected domain and the boundary condition is $u \cdot \nu = 0$ on $\partial\Omega$. Alternatively, Ω could be the flat torus. Now let $u_0(x)$ be the velocity field of a smooth equilibrium flow (of which there are many possibilities).

Linearization #1. The straightforward linearization is

$$(\partial_t + u_0 \cdot \nabla)v + (v \cdot \nabla)u_0 + \nabla q = 0, \qquad \nabla \cdot v = 0.$$

It generates a semigroup in $L^2(\Omega)$ with a generator L_1. The operator $v \to (v \cdot \nabla)u_0$ is not compact. Its *essential spectrum* is governed by a system of ODEs, as follows.

$$\begin{cases} \dot{x} = u_0(x) \\ \dot{\xi} = -(\nabla u_0)^T \xi \\ \dot{b} = -(\nabla u_0)b + 2(\xi \cdot \nabla u_0)b\,\xi/|\xi|^2. \end{cases}$$

The orbits are generalized bicharacteristics. Indeed, Friedlander and Vishik (1992) proved for the torus that the essential spectral radius of e^{tL_1} equals the maximum growth rate of the ODE orbits. In particular, if the orbits grow exponentially, then e^{tL_1} has some essential spectrum in the exterior of the closed unit disk. The fluid is "stretched" along streamlines.

Linearization #2. Consider the vorticity $\omega = \text{curl } u = \partial_1 u_2 - \partial_2 u_1$. It satisfies

$$(\partial_t + u \cdot \nabla)\omega = 0$$

because $n = 2$. Linearizing this equation, and using the notation $\eta = \delta\omega,\ v = \delta u$, we have

$$(\partial_t + u_0 \cdot \nabla)\eta + \nabla(\nabla \times u_0) \cdot v = 0, \qquad \nabla \cdot v = 0.$$

Acting on the linearized vorticity η, the generator therefore is $L_2 = -u_0 \cdot \nabla - \nabla(\nabla \times u_0) \cdot (\text{curl})^{-1}$. Considering L_2 acting on $L^2(\Omega)$ is roughly equivalent to considering L_1 acting on $H^1(\Omega)$, because $n = 2$ and $\nabla \cdot v = 0$.

However, in contrast to e^{tL_1}, the essential spectrum of e^{tL_2} has no growth because $-u_0 \cdot \nabla$ is skew-adjoint and the second term in L_2 is a compact operator. Thus the essential spectrum is stable, in the sense that if $\lambda \in \text{ess spec}(L_2)$, then $\Re\lambda = 0$. So instability in the sense of the second linearization could only occur in the discrete spectrum. The following theorem [Bardos-Guo-S 2002] (for the case of a bounded domain) relates this kind of linear instability to nonlinear instability.

Theorem 3.3. *If L_2 has point spectrum $\Re\lambda > \sigma$ (with σ given below), then u_0 is nonlinearly unstable in the space $H^1(\Omega)$. (That is, the space $u \in H^1, \omega \in L^2$.)*

The space is the one for which Arnold's stability theorem is valid. Here σ is the *classical growth rate* for the ODE $\dot{x} = u_0(x)$. That is, if $X(t,x)$ denotes the flow for this ODE, then

$$\sigma = \sup_x \lim_{t\to+\infty} \frac{1}{t} \log \left| \frac{\partial X}{\partial x} \right|.$$

In this theorem there is no further restriction on the domain.
Exercise: For a parallel shear flow show that $\sigma = 0$.

Here is a brief sketch of the proof of the theorem. Write $\omega_0 = \operatorname{curl} u_0$. The *full* nonlinear equation $(\partial_t + u \cdot \nabla)\omega = 0$ is rewritten as

$$(\partial_t + u_0 \cdot \nabla)(\omega - \omega_0) + (u - u_0) \cdot \nabla\omega_0 = -(u - u_0) \cdot \nabla(\omega - \omega_0)$$

or, for brevity,

$$(\partial_t - L_2)(\omega - \omega_0) = Q, \qquad L_2 = -(\operatorname{curl}^{-1}\omega_0) \cdot \nabla - \nabla\omega_0 \cdot \operatorname{curl}^{-1}.$$

We choose the perturbation in the most unstable possible direction $\omega(0,x) = \omega_0 + \delta\chi(x)$ where $L_2\chi = \lambda\chi$ and $\Re\lambda > \sigma$ is maximal. For simplicity of exposition, suppose λ is real. Now we rewrite the above PDE as

$$\omega(t) - \omega_0 = \delta e^{\lambda t}\chi + \int_0^t e^{(t-s)L_2} Q(s) ds.$$

Taking L^2 norms, we have the inequality

$$\|\omega(t) - \omega_0\|_{L^2} \geq \delta e^{\lambda t} - c_\epsilon \int_0^t e^{(t-s)(\lambda+\epsilon)} \|u(s) - u_0\|_{L^\infty} \|\nabla(\omega(s) - \omega_0)\|_{L^2} ds.$$

We would like to show that the whole integral is less than $c[\delta\exp(\lambda t)]^2$. Were that the case, we could deduce that $\|\omega(t) - \omega_0\|_{L^2} > \epsilon_0 > 0$ provided we stay within a time interval where $\delta\exp(\lambda t)$ is sufficiently small.
Exercise: Justify this statement.

Lemma 3.4 (Classical Flows)**.** *Given $u_0(x)$ with $\nabla \cdot u_0 = 0$ as well as $v(t,x)$ with $\nabla \cdot v = 0$, denote*

$$\partial_t X_0 = u_0(X_0), \quad X_0(0) = x, \qquad \partial_t X = v(t,X), \quad X(0) = x.$$

If $\|v(t) - u_0\|_{C^1} < \delta e^{\lambda t}$ with $\lambda > \sigma + \epsilon$, then

$$\left| \frac{\partial X}{\partial x} - \frac{\partial X_0}{\partial x} \right| \leq c_\epsilon \theta_0 e^{(\sigma+\epsilon)t}$$

in a short time-interval $(0,T^*)$ with $\delta e^{\lambda t} \leq \theta_0$ sufficiently small.

Proof. Denote $A_0 = \partial_x(X_0)$, a 2×2 matrix. If we differentiate $\partial_t X_0 = u_0(X_0)$ with respect to x, we find $\partial_t(\partial X_0/\partial x) = A_0(\partial X_0/\partial x)$. Consider the difference $X - X_0$. By a Taylor expansion,

$$\begin{aligned}(\partial_t - A_0)(X - X_0) &= v(t,X) - u_0(X_0) - (\partial_x u_0)(X_0)(X - X_0) \\ &= \frac{1}{2}\partial_x^2 u_0(\bar{x})(X - X_0)^2 + v(t,X) - u_0(X) \leq O\{|X - X_0|^2\} + \delta e^{\lambda t}.\end{aligned}$$

Hence $|X - X_0| \le c\delta e^{\lambda t}$ in an interval $(0, T^*)$ where $\delta e^{\lambda t}$ is sufficiently small. Now $(\partial_x v)(t, X) - A_0$ is also small because both $\|v - u_0\|_{C^1}$ and $X - X_0$ are small. Next, let $Y = \partial_{x_j}(X - X_0)$ for $j = 1, 2$. Then by differentiation,

$$(\partial_t - A_0)Y = \{(\partial_x v)(X) - A_0\}\,(\partial X_0/\partial x_j + Y)\,.$$

This is a linear equation in Y with a small coefficient on the right side. Moreover, A_0 induces growth no worse than $\exp[(\sigma + \epsilon/2)t]$. Hence

$$|Y(t)| \le c\theta_0 e^{(\sigma+\epsilon)t} \quad \text{in } (0, T^*). \qquad \square$$

□

Corollary 3.5. *If $\|v(t) - u_0\|_{C^1} \le \delta \exp{(\sigma + \epsilon)t}$, then any solution of the equation $(\partial_t + v \cdot \nabla)h = 0$ satisfies the estimate*

$$\|h(t)\|_{W^{1,p}} \le c_\epsilon e^{(\sigma+\epsilon)t} \|h(0)\|_{W^{1,p}}$$

within the same time-interval $(0, T^*)$, for any p.

Proof. The solution h is given by characteristics as $h(t, x) = h(0, X(t, x))$. The estimate follows immediately from the preceding lemma. □

Theorem 3.6 (Bootstrap)**.** *Let $p > 2$. If a solution of the Euler equation satisfies $\|\omega(0) - \omega_0\|_{W^{1,p}} \le c\delta$ and $\|\omega(t) - \omega_0\|_{L^2} \le c\delta e^{\lambda t}$ in some time interval $\supset (0, T^*)$ with $\lambda > \sigma + \epsilon$, then*

$$\|\omega(t) - \omega_0\|_{W^{1,p}} \le c\delta e^{(\sigma+\epsilon)t} \quad \textit{in } (0, T^*).$$

Proof. This time we rewrite the full nonlinear equation for the vorticity as

$$(\partial_t + u \cdot \nabla)(\omega - \omega_0) = -(u - u_0) \cdot \nabla\omega_0.$$

Note that $W^{1,p} \subset C$ for $p > 2$. We use the corollary to estimate

$$\|\omega(t) - \omega_0\|_{W^{1,p}} \le c\delta e^{(\sigma+\epsilon)t} + c\int_0^t e^{(\sigma+\epsilon)(t-s)}\ \|(u(s) - u_0) \cdot \nabla\omega_0\|_{W^{1,p}}\ ds.$$

The norm inside the integral is estimated by

$$\|u(t) - u_0\|_{C^1} \le c\|u(t) - u_0\|_{W^{2,p}} \le c\|\omega(t) - \omega_0\|_{W^{1,p}}.$$

This closes the loop that $\|u(t) - u_0\|_{C^1} \le \delta e^{\lambda t}$. Adjust ϵ and put this result back into the main inequality. See [BGS] for details. □

4 Linear to nonlinear instability

For simplicity we'll take $\phi = 0$. The equation is written as

$$du/dt = Lu + F(u)$$

where $L = A'(\phi)$ and $F(u) = O(|u|^2)$ formally.

A precise formulation of the main question about instability is as follows. Assume that we have
(i) two Banach spaces $X \subset Z$,
(ii) a strongly continuous semigroup e^{tL} on Z, and
(iii) a nonlinear operator $F : X \to Z$ that satisfies $\|F(u)\|_Z \le c\|u\|_X^\alpha \|u\|_Z^\beta$ for $\|u\|_X$ small, where $\beta > 1$ and $\alpha \ge 0$.

Fundamental question: If spec(e^{tL}) meets the exterior of the closed unit disc, is $u = 0$ (nonlinearly) unstable in X? The next three theorems give an affirmative answer under three conditions.

Theorem 4.1 (Grillakis-Shatah-S 1990). *True if there exists some point spectrum $e^{\lambda_0 t}$ "near" the maximal growth of e^{tL}. More precisely,*

$$\Re\lambda_0 > \frac{1}{\beta} \lim_{t\to+\infty} \frac{1}{t} \log \|e^{tL}\|_Z.$$

Theorem 4.2 (Friedlander-S-Vishik 1997). *True if there exists a spectral gap outside the unit disk. This means there exists an annulus outside the unit disk that is entirely within the resolvent set of e^L.*

Theorem 4.3 (Shatah-S 2000). *True if $X = Z$.*

The following "spectral dangers" occur in these theorems.
1°. $\sigma(e^{tL}) \supset e^{\sigma(tL)}$ but not necessarily =. (violation of the spectral mapping theorem)
2°. It is possible that $\|e^{tL}\|$ is greater than the spectral radius of e^{tL}.
3°. e^{tL} could have continuous or residual spectrum.

For many interesting PDEs these theorems do not apply precisely but their basic ideas do. We shall now discuss the last of these theorems in detail.

4.1 Instability theorem

Consider a Banach space X and the evolution equation

$$\frac{du}{dt} = Lu + F(u) \qquad (u(t) \in X) \tag{4.1}$$

where L is a linear operator that generates a strongly continuous semigroup $\exp(tL)$, and F is a strongly continuous operator such that $F(0) = 0$. By a *solution* of (4.1) in an interval I, we mean that $u \in C(I; X)$ satisfies the integral equation that is associated to (4.1). The question we address is the following.

If the spectrum of L meets the right half-plane $\{\Re\lambda > 0\}$, does it follow that the zero solution ($u \equiv 0$) is unstable?

The zero solution is called *nonlinearly stable* if $\forall\epsilon > 0\ \exists\delta > 0$ such that if $\|u_0\| < \delta$, then there is a unique solution $u \in C([0,\infty); X)$ of equation (4.1) with $u(0) = u_0$ such that

$$\sup_{0\le t<\infty} \|u(t)\| < \epsilon.$$

Otherwise, it is called *nonlinearly unstable.* We restate the last theorem as

Theorem 4.4. *Assume the following.*

(i) L generates a strongly continuous semigroup on a Banach space X.

(ii) The spectrum of L meets the right half-plane $\{\Re\lambda > 0\}$.

(iii) $F : X \to X$ is continuous and $\exists\rho_o > 0, \eta > 0$ and $c_2 > 0$ such that $\|F(u)\| \le c_2\|u\|^{1+\eta}$ for $\|u\| < \rho_o$.

Then the zero solution is nonlinearly unstable.

In fact, the proof will show that hypothesis (ii) can be replaced by the weaker hypothesis (ii') *The spectrum of e^L meets the exterior $\{|\lambda| > 1\}$ of the unit disk.*

Lemma 4.5. *If B is a closed linear operator and μ lies on the boundary of the spectrum of B, then μ belongs to the approximate point spectrum of B. That is,*

$$\inf_{v\in D(B),\|v\|=1} \|(B-\mu)v\| = 0.$$

Proof. We denote by $\sigma(C)$ the spectrum of any operator C. Let $\mu \in \partial(\sigma(B))$. Let $\mu_n \to \mu$ where $\mu_n \notin \sigma(B)$. Thus $(\mu_n - B)^{-1}$ exists. Its norm is at least as large as its spectral radius. Its spectrum is $\sigma((\mu_n - B)^{-1}) = (\mu_n - \sigma(B))^{-1}$. Hence

$$\|(\mu_n - B)^{-1}\| \ge \frac{1}{\operatorname{dist}(\mu_n, \sigma(B))} \to \infty.$$

By the Uniform Boundedness Principle, there exists $v \in X$ such that $\|(\mu_n - B)^{-1}v\| \to \infty$. Let

$$v_n = \frac{(\mu_n - B)^{-1}v}{\|(\mu_n - B)^{-1}v\|}.$$

Then $v_n \in D(B)$ and $\|v_n\| = 1$. Furthermore, we apply the identity

$$(B-\mu)(\mu_n - B)^{-1} = (\mu_n - \mu)(\mu_n - B)^{-1} - I$$

to the vector $v\|(\mu_n - B)^{-1}v\|^{-1}$. We obtain

$$\|(B-\mu)v_n\| = \left\|(\mu_n - \mu)v_n - \frac{v}{\|(\mu_n - B)^{-1}v\|}\right\| \to 0.$$

Thus we have $v_n \in D(B)$, $\|v_n\| = 1$ and $\|Bv_n - \mu v_n\| \to 0$. $\square$

Lemma 4.6. *Let $e^\lambda \in \sigma(e^L)$ such that $|e^\lambda|$ equals the spectral radius of e^L. For every $\gamma > 0$ and every integer $m > 0$, there exists $v \in X$ such that*

$$\|(e^{mL} - e^{m\lambda})v\| < \gamma\|v\| \tag{4.2}$$

and

$$\|e^{tL}v\| \le 2Ke^{t\Re\lambda}\|v\| \quad \forall\ 0 \le t \le m \tag{4.3}$$

where $K = \sup\{\|\exp(\theta L)\| :\ 0 \le \theta \le 1\}$.

Proof. It is well-known that in general $\sigma(e^L) \supset e^{\sigma(L)}$. (However, these two sets are not necessarily equal.) Therefore hypothesis (ii) implies hypothesis (ii'). By (ii'), the operator e^L has some spectrum μ outside the unit disk. We choose μ to belong to the outer boundary of $\sigma(e^L)$; that is, $|\mu| = \max\{|\nu| : \nu \in \sigma(e^L)\}$.

Now we apply Lemma 4.5 to $B = e^L$ and write $\mu = e^\lambda$. Thus there exists a sequence $\|v_n\| = 1$ such that $(e^L - e^\lambda)v_n \to 0$. It follows that, for all integers $m > 0$,

$$(e^{mL} - e^{m\lambda})v_n = \sum_{j=0}^{m-1}[e^{jL} - e^{(m-1-j)\lambda}][e^L - e^\lambda]v_n \to 0$$

as $n \to \infty$. Given $\gamma > 0$, we choose n so large that v_n satisfies both (3) and $\|(e^{jL} - e^{j\lambda})v_n\| < 1$ for $j = 0, 1, ..., m$. Now let $0 \le t \le m$ and let $j = [t]$, the greatest integer $\le t$. Then

$$\|e^{tL}v_n\| \le K\|e^{jL}v_n\| \le K(\|e^{j\lambda}v_n\| + 1) < 2Ke^{t\Re\lambda}.$$

Thus (4) is also valid and the lemma is proven.

□

Lemma 4.7. *Let λ be as above. For all $\epsilon > 0$ there exists C_ϵ so that for all $0 \le t < \infty$ we have*

$$e^{(\Re\lambda)t} \le \|e^{tL}\| \le C_\epsilon e^{(\Re\lambda+\epsilon)t}. \tag{4.4}$$

Proof. By the definition of λ, the spectral radius of e^L is

$$e^{\Re\lambda} = \lim_{m\to\infty} \|e^{mL}\|^{1/m}.$$

Therefore there exists a time S_ϵ such that

$$e^{\Re\lambda-\epsilon} < \|e^{mL}\|^{1/m} < e^{\Re\lambda+\epsilon}$$

for every integer $m \ge S_\epsilon - 1$. Now let $t > S_\epsilon$ and let $m = [t]$. Then

$$\|e^{tL}\| \le K\|e^{mL}\| < Ke^{(\Re\lambda+\epsilon)m} \le Ke^{(\Re\lambda+\epsilon)t}$$

and also

$$K\|e^{tL}\| \ge \|e^{(m+1)L}\| > e^{(\Re\lambda-\epsilon)(m+1)} \ge e^{(\Re\lambda-\epsilon)t}.$$

Thus

$$K^{-1}e^{(\Re\lambda-\epsilon)t} < \|e^{tL}\| < Ke^{(\Re\lambda+\epsilon)t} \quad \forall t > S_\epsilon.$$

Since $\|e^{tL}\|$ is bounded for $0 \le t \le S_\epsilon$, the upper bound in (5) is valid for some constant C_ϵ.

Now given any $t > 0$, let N be any integer larger than S_ϵ/t. Then

$$K^{-1}e^{(\Re\lambda-\epsilon)Nt} < \|e^{NtL}\| \le \|e^{tL}\|^N.$$

Taking Nth roots, we have $K^{-1/N}e^{(\Re\lambda-\epsilon)t} < \|e^{tL}\|$. Letting $N \to \infty$ and then $\epsilon \to 0$, we obtain

$$\|e^{tL}\| \ge e^{\Re\lambda t}.$$

□

Proof. (of Theorem 4.4)

We are given $\mu = e^\lambda$ as in Lemma 4.6. For a fixed number k to be defined later, let $0 < \delta < \min\{k^{-1}, \rho_o/2, 1\}$. The parameter δ will be free to remain arbitrarily small. Choose T^*, depending on δ, to be the positive *integer* defined by

$$\tfrac{1}{k} < \delta e^{T^*\Re\lambda} \le \tfrac{|\mu|}{k}. \tag{4.5}$$

Indeed, we can choose T^* to belong to the interval $(b, b+1]$ where $b = \ln(1/\delta k)/\ln|\mu| > 0$.

Furthermore, let v be given by Lemma 2 with $m = T^*$ and $\gamma = (4k)^{-1}$. We normalize $\|v\| = \delta$. Thus from (4.2) and (4.5) we have

$$\|e^{T^*L}v\| > \|e^{T^*\lambda}v\| - \tfrac{\delta}{4k} > \tfrac{1}{k} - \tfrac{\delta}{4k} \tag{4.6}$$

and from (4.3) we have

$$\|e^{tL}v\| \le 2K\delta e^{t\Re\lambda} \quad \forall\ 0 \le t \le T^*. \tag{4.7}$$

Now we begin to look at the full nonlinear equation. By contradiction, suppose that the zero solution is stable. Thus there exists $\delta_0 > 0$ such that if $\|v\| = \delta < \delta_0$, then there exists a unique solution $u \in C([0,\infty); X)$ of the integral equation

$$u(t) = e^{tL}v + \int_0^t e^{(t-\tau)L}\ F(u(\tau))\ d\tau.$$

It would suffice to prove that $\|u(t)\| > \frac{1}{4k}$ at some time t, for that would contradict the stability.

By hypothesis (iii) on the nonlinear operator we have

$$\|u(t) - e^{tL}v\| \le \int_0^t \|e^{(t-\tau)L}\|\ c_2\|u(\tau)\|^{1+\eta}\ d\tau$$

provided $\|u(\tau)\| < \rho_o$ in the interval $[0,t]$. By (4.4), we estimate

$$\|u(t) - e^{tL}v\| \le c_1c_2 \int_0^t e^{(1+\eta/2)\Re\lambda(t-\tau)}\|u(\tau)\|^{1+\eta}\ d\tau \tag{4.8}$$

where $\epsilon = \Re\lambda\eta/2$ and $c_1 = C_\epsilon$. Now define

$$T = \sup\left\{t : \|u(\tau) - e^{\tau L}v\| < \tfrac{1}{2|\mu|}\delta e^{\Re\lambda\tau} \quad\text{and}\quad \|u(\tau)\| < \tfrac{\rho_o}{2} \quad\text{for } 0 \le \tau \le t\right\}. \tag{4.9}$$

Clearly $T > 0$. For $t \le \min\{T, T^*\}$ we have by (4.8), (4.9) and (4.7),

$$\begin{aligned}
&\|u(t) - e^{tL}v\| \\
&\le c_1c_2 \int_0^t e^{(1+\eta/2)\Re\lambda(t-\tau)} \left(\|e^{\tau L}v\| + \|u(\tau) - e^{\tau L}v\|\right)^{1+\eta}\ d\tau \\
&\le c_1c_2 \int_0^t e^{(1+\eta/2)\Re\lambda(t-\tau)} \left(2K\delta e^{\Re\lambda\tau} + \tfrac{1}{2|\mu|}\delta e^{\Re\lambda\tau}\right)^{1+\eta}\ d\tau \\
&\le c_1c_2 \left(2K + \tfrac{1}{2|\mu|}\right)^{1+\eta} \delta^{1+\eta} e^{(1+\eta/2)\Re\lambda t} \int_0^t e^{(\eta/2)\Re\lambda\tau}\ dt \\
&< c_1c_2 \left(2K + \tfrac{1}{2|\mu|}\right)^{1+\eta} \delta^{1+\eta} \tfrac{2}{\eta\Re\lambda} e^{(1+\eta)\Re\lambda t} \\
&\equiv \tfrac{k^\eta}{2|\mu|^{1+\eta}} (\delta e^{\Re\lambda t})^{1+\eta}
\end{aligned}$$

where k is defined as

$$k^\eta = 2|\mu|^{1+\eta} c_1c_2(2K + \tfrac{1}{2|\mu|})^{1+\eta} \tfrac{2}{\eta\Re\lambda}.$$

We claim that either $T^* < T$ or else $\|u(T)\| = \rho_o/2$. Indeed, assume on the contrary that $T \le T^*$ and $\|u(T)\| < \rho_o/2$. Then using (4.9) in our estimate with $t = T$, we have

$$\tfrac{1}{2|\mu|}\delta e^{\Re\lambda T} = \|u(T) - e^{TL}v\| < \tfrac{k^\eta}{2|\mu|^{1+\eta}} (\delta e^{\Re\lambda T})^{1+\eta}.$$

Thus

$$(\delta e^{\Re\lambda T})^\eta > \left(\tfrac{|\mu|}{k}\right)^\eta \ge (\delta e^{\Re\lambda T^*})^\eta$$

by (4.5). So $T > T^*$, a contradiction. This proves the claim.

Assuming that $\|u(T)\| \ne \rho_o/2$, we have $T^* < T$ so that we may put $t = T^*$ in our estimate to obtain

$$\|u(T^*) - e^{T^*L}v\| < \frac{k^\eta}{2|\mu|^{1+\eta}} \left(\delta e^{\Re\lambda T^*}\right)^{1+\eta} \le \frac{k^\eta}{2|\mu|^{1+\eta}} \left(\frac{|\mu|}{k}\right)^{1+\eta} = \frac{1}{2k}. \tag{4.10}$$

It follows from (4.10) and (4.6) that

$$\|u(T^*)\| > \|e^{T^*L}v\| - \frac{1}{2k} > \frac{1}{2k} - \frac{\delta}{4k} > \frac{1}{4k}$$

since $\delta < 1$. It follows in any case that there is a time t (either T or T^*) at which $\|u(t)\| \ge \min\{\frac{1}{4k}, \frac{\rho_o}{2}\} \equiv \epsilon_o$. This completes the proof. $\square$

Remark. The proof shows that there exist $C > 0$ and $\epsilon_0 > 0$ such that for all sufficiently small positive δ, there is a solution u that satisfies $\|u(0)\| < \delta$ but $\sup_{0\le t\le C|\log\delta|} \|u(t)\| \ge \epsilon_0$. Thus the escape time occurs logarithmically soon.

4.2 Application to a regularized Boussinesq Equation

In this section we shall apply the abstract theorem to a particular problem where e^L has some unstable point spectrum. We do not know whether there is any point spectrum near the spectral boundary, nor do we know whether there is a gap in the spectrum. The regularized Boussinesq equation (RB) is

$$u_{tt} - u_{xxtt} - u_{xx} - f(u)_{xx} = 0$$

which we take for $-\infty < x < +\infty$. A solitary wave solution $u = \varphi(x - ct)$, where φ vanishes appropriately at infinity, must satisfy the equation

$$-\varphi'' + (1 - c^{-2})\varphi - c^{-2} f(\varphi) = 0.$$

We assume that $f(u) = |u|^{p-1}u$ where $1 < p < \infty$. The sign is important. If $c^2 > 1$, there exists a solitary wave solution which is positive, exponentially decaying, and even.

Theorem 4.8. *If $p > 5$ and $1 < c^2 < \frac{3(p-1)}{2(p+1)}$, then the solitary wave $\varphi(x - ct)$ is nonlinearly unstable in the space $H^1 \times H^2$.*

Proof. An explanation of this space is in order. It is convenient to denote $\partial = \partial/\partial x$ and write the equation in the equivalent form

$$u_t = v_x, \qquad v_t = \partial(1 - \partial^2)^{-1}\{u + f(u)\}.$$

We denote $\mathbf{u} =$ the pair $[u, v]^T$. By $H^s \times H^{s+1}$ we mean the space of all pairs such that $u \in H^s(\mathbb{R})$ and $v \in H^{s+1}(\mathbb{R})$. Global existence and uniqueness are known [Y. Liu] in this space for any $s \geq 1$. The invariants of the equation are

$$E = \frac{1}{2}\int [u^2 + v^2 + v_x^2 + 2F(u)]\ dx, \quad Q = \int [uv + u_x v_x]\ dx, \quad I = \int u\ dx$$

where $F' = f$. Thus the natural norm associated with the energy E is $L^2 \times H^1$. For the f that we are considering, $F(u) = |u|^{p+1}/(p + 1) \geq 0$, so that E is positive and we have global estimates.

For convenience, we could rewrite the equation in the Hamiltonian form

$$\frac{d\mathbf{u}}{dt} = JE'(\mathbf{u})$$

where

$$J = (1 - \partial^2)^{-1}\partial \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad E'(\mathbf{u}) = \begin{pmatrix} u + f(u) \\ (1 - \partial^2)v \end{pmatrix}.$$

Furthermore, we let

$$\boldsymbol{\varphi} = \begin{pmatrix} \varphi \\ -c\,\partial\varphi \end{pmatrix} \quad \text{and} \quad H = E''(\boldsymbol{\varphi}) + cQ''(\boldsymbol{\varphi}) = \begin{pmatrix} 1 + f'(\varphi) & c(1 - \partial^2) \\ c(1 - \partial^2) & 1 - \partial^2 \end{pmatrix}.$$

Note that the linearized Hamiltonian H has its essential spectrum unbounded both to the left and to the right. Now the change of variables $\mathbf{u}(x,t) = \boldsymbol{\varphi}(x-ct) + \mathbf{w}(x-ct,t)$ leads to the equation

$$\frac{d\mathbf{w}}{dt} = L\mathbf{w} + F(\mathbf{w})$$

where

$$L = JH = A + K = \begin{pmatrix} c\partial & \partial \\ \partial(1-\partial^2)^{-1} & c\partial \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \partial(1-\partial^2)^{-1} f'(\varphi) & 0 \end{pmatrix}$$

is the linearized generator and

$$F(\mathbf{w}) = F\begin{pmatrix} w \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \partial(1-\partial^2)^{-1}[f(\varphi+w) - f(\varphi) - f'(\varphi)w] \end{pmatrix}$$

is the nonlinear operator.

We apply Theorem 4.4 to the equation and the space $X = H^1 \times H^2$. On this space X, the operator L generates a strongly continuous semigroup because A is skew-adjoint and K is a bounded operator. Furthermore, it is known [Pego-Weinstein 1992] using the Evans function that, under the assumptions of Theorem 4.8, L has an eigenvalue (point spectrum) λ with $\Re\lambda > 0$. Thus conditions (i) and (ii) of Theorem 1 are satisfied.

It remains to check condition (iii). We have

$$\begin{aligned} \|F(\mathbf{w})\| &= \|\partial(1-\partial^2)^{-1}[f(\varphi+w) - f(\varphi) - f'(\varphi)w]\|_{H^2} \\ &\le \|f(\varphi+w) - f(\varphi) - f'(\varphi)w\|_{H^1} \\ &\le C\|(|\varphi|^{p-2} + |w|^{p-2})w^2\|_{H^1} \\ &\le C(\|w\|_{H^1}^2 + \|w\|_{H^1}^p) \\ &\le C'\|\mathbf{w}\|_X^2 \end{aligned}$$

for $\|\mathbf{w}\|_X^2 \le 1$ since $H^1 \subset L^\infty$ and $p \ge 2$. Therefore all the conditions of Theorem 4.4 are satisfied, so that the nonlinear instability is proven. □

According to the remark at the end of the previous section, the escape time occurs logarithmically soon. This excludes the trivial apparent instability due to a solitary wave traveling at a slightly different speed c', which merely separates linearly soon because $\|\boldsymbol{\varphi}_{c'}(x-c't) - \boldsymbol{\varphi}_c(x-ct)\| = O(t)$. Orbital instability is also true but we omit the proof.

4.3 Application to the Kuramoto-Sivashinsky equation

The equation is

$$u_t + u_{xxxx} + u_{xx} + uu_x = 0.$$

It is a one-dimensional model in the theory of flame propagation. There are many numerical and some theoretical results showing that some of its solutions engage in very complicated dynamical behavior. It has many traveling wave solutions $u = \phi(x-ct)$ for which the two limits

$$b_\pm = \lim_{t\to\pm\infty} \phi(\xi)$$

exist [Troy 1989]. These $\phi(\xi)$ have multiple maxima and minima. The following theorem [Wang-S 2002] asserts their instability.

Theorem 4.9. *Any such traveling wave is nonlinearly unstable under $H^1(\mathbb{R})$ perturbations. That is, there exists an $\epsilon_0 > 0$ and a family of solutions $u^\delta(t,x)$ such that*

$$\|u^\delta(0,\cdot) - \phi(\cdot)\|_{H^1} < \delta \qquad (0 < \delta \le \delta_0)$$

but

$$\sup_{0 \le t \le C|\log\delta|} \|u^\delta(t,\cdot) - \phi(\cdot - ct)\|_{H^1} > \epsilon_0.$$

A traveling wave solution $u = \phi(x - ct)$ satisfies, after one integration, the third-order equation

$$\phi''' + \phi' + \frac{1}{2}(\phi - c)^2 = k \tag{4.11}$$

where k is a constant. A special case is a steady state $c = 0$. This ordinary differential equation has been studied extensively. Numerical studies indicate the existence of heteroclinic and homoclinic orbits, as well as periodic and quasiperiodic solutions. Theoretical results include the existence of periodic solutions and heteroclinic orbits . In particular, Troy proved that if $k = 1$, there exist at least two distinct odd solutions of (4.11) such that $\phi(x) \to c \mp \sqrt{2}$ as $x \to \pm\infty$. He conjectured that there are an infinite number of different ones. Furthermore for $k \neq 1$ there are probably many others.

We will show that the essential spectrum of the linearized generator meets the right half-plane and thus generates modes $e^{\lambda t}$ with $\mathcal{R}\lambda > 0$. In fact, we write the linearized generator after translation as

$$L = L_0 - (\phi - b_+ - c)\partial - \phi_x, \text{ where } L_0 = -\partial^4 - \partial^2 - b_+\partial.$$

Assume $b_- = 0$. By Fourier transformation, L_0 has unstable essential spectrum while the last two terms define a relatively compact operator. In this way we will prove that e^{tL} has essential spectrum outside the closed unit disk.

Note that the nonlinear term uu_x is *not* a bounded operator on H^1. However, a generalization of Theorem 4.4 allows two different Banach spaces $X \subset Z$, the linear semigroup is *smoothing* (mapping Z into X), while the nonlinear term *loses regularity* (mapping X into Z). The gain and loss of regularity compensate for each other.

Theorem 4.10. *Assume the following.*

(i) X, Z are two Banach spaces with $X \subset Z$ and $\|u\|_Z \le C_1\|u\|_X$ for $u \in X$.

(ii) L generates a strongly continuous semigroup e^{tL} on the space Z, and the semigroup e^{tL} maps Z into X for $t > 0$, and $\int_0^1 \|e^{tL}\|_{Z\to X}\, dt = C_4 < \infty$.

(iii)The spectrum of e^L meets the exterior of the unit disk.

(iv)$F : X \to Z$ is continuous and $\exists \rho_0 > 0, C_3 > 0, \alpha > 1$ such that $\|F(u)\|_Z \le C_3\|u\|_X^\alpha$ for $\|u\|_X < \rho_0$.

Then the zero solution of (1.1) is nonlinearly unstable in the space X.

Long **exercise**: Prove it, generalizing Theorem 4.4.

We take $Z = L^2(\mathbb{R})$ and $X = H^1(\mathbb{R})$. $F(w) = -ww_x$. Note that F maps $H^1(\mathbb{R})$ into $L^2(\mathbb{R})$ and satisfies

$$\|F(w)\|_{L^2} \le \|w\|^2_{H^1}.$$

This is Condition (iv) with $\alpha = 2$. The following lemma proves Condition (ii).

Lemma 4.11. *Let* $\phi \in L^\infty(\mathbb{R}), \quad \phi_x \in L^2(\mathbb{R}), \quad \chi_{[0,\infty)}(\phi - b_+) \in L^2(\mathbb{R})$. *Then*

$$\|e^{tL}\|_{L^2\to H^1} \le Ct^{-\frac{1}{4}}, \qquad for \quad 0 < t \le 1,$$

$$\|e^{tL}\|_{H^1\to H^1} \le C < \infty, \qquad for \quad 0 \le t \le 1,.$$

Proof. By Fourier transformation, the essential spectrum of L_0 on $H^1(\mathbb{R})$ is

$$\sigma_e(L_0) \supset \{-\xi^4 + \xi^2 - ib_+\xi \quad : \quad \xi \in \mathbb{R}\}.$$

This curve meets the vertical lines $Re\lambda = \alpha$ for $-\infty < \alpha \le \frac{1}{4}$ because $-\infty < -\xi^4 + \xi^2 \le \frac{1}{4}$. We then prove that the same curve belongs to the essential spectrum of L. This implies Condition (iii).
Exercise: Use this to prove the lemma, first for L_0 and then for L. □

5 General theory of solitary waves

5.1 General framework for solitary wave solutions of a Hamiltonian system

Following [Grillakis-Shatah-S 1987,1990], we consider

$$\frac{du}{dt} = JE'(u(t)), \quad u(t) \in X, \tag{5.1}$$

where X is a Hilbert space, J is a closed linear operator on X which is skew ($J^* = -J$), and $E : X \to \mathbb{R}$ is twice Frechet-differentiable with derivative E'. Furthermore, there is a group G of transformations acting unitarily on X, meaning that there is a mapping T from G to unitary operators on X. We assume that G leaves the equation invariant. Its derivative, denoted by T'_ω , is a skew-adjoint operator on X.

Definition. A "solitary wave" is a solution of the special form $u(t) = T(\exp \omega t)\phi$, where $\omega \in \mathfrak{g}$, the Lie algebra, and $\phi \in X$. Linearizing around such a solitary wave, we get the equation

$$\frac{dv}{dt} = JHv(t), \quad H = E''(\phi) - Q''_\omega(\phi), \quad Q_\omega(\phi) = \frac{1}{2}\langle J^{-1}T'_\omega\phi, \phi\rangle \tag{5.2}$$

Thus JH is the linearized generator. Although H is self-adjoint and J is skew-adjoint, these two operators *do not commute*. The spectrum of JH is symmetric across both axes $\mathbb{R}$ and $i\mathbb{R}$. So linearized stability would mean that all of its spectrum belongs to $i\mathbb{R}$. However, our goal is to find verifiable conditions for stability on the simpler operator H.

Let's specialize to $G = \mathbb{R}$ under addition, with $\mathfrak{g}$ also identified as $\mathbb{R}$. Then a "solitary wave" is $u(t) = T(\omega t)\phi$ with $\omega \in \mathbb{R}$. Invariance means that

$$T(s+r) = T(s)T(r), \quad \|T(s)u\| = \|u\|, \quad E(T(s)u) = E(u), \quad T(s)J = JT^*(-s).$$

The generator of the group $T(\cdot)$ of unitary operators is denoted $T'(0)$. $T'(0)$ is a skew-adjoint operator on X with a dense domain $D(T'(0))$. Let's assume that J is skew (on its domain) and 1-1 onto (this can be generalized). Assume that $J^{-1}T'(0)$ extends to a bounded operator on X. Another invariant is

$$Q(v) = \frac{1}{2}\langle J^{-1}T'(0)v, v\rangle.$$

Exercise: Show that $Q(T(s)v) = Q(v)$ for all $v \in X,\ s \in \mathbb{R}$.
Exercise: Show that, formally, for a solution of (5.1), $E(u(t))$ and $Q(u(t)$ are independent of t.

Example 1: Smooth peakons with $k > 0$.

$$u_t - u_{txx} + 3uu_x + 2ku_x = 2u_x u_{xx} + uu_{xxx} \tag{5.3}$$

Look for a traveling wave solution $u(x,t) = \phi(x - \omega t)$, $\omega =$ constant, decaying at ∞. Get the ODE $(\omega - \phi)\phi_x^2 = (\omega - 2k - \phi)\phi^2$. For $\omega > 2k$, there exists a peakon that looks like the one for $k = 0$ except that it is *smooth*.
Exercise: Explain why the previous proof fails for $k > 0$, while the proof to follow fails for $k = 0$.

The group $G = \mathbb{R}$ acts on $X = H^1(\mathbb{R})$ by $u(x) \to u(x - s)$ (translation). The skew operator is $J = \frac{1}{2}\partial(1 - \partial^2)^{-1}$. The invariants are

$$Q(u) = -\int_{\mathbb{R}} (u^2 + u_x^2)dx, \quad E(u) = -\int_{\mathbb{R}} (u^3 + uu_x^2 + 2ku^2)dx.$$

Exercise: Verify that the solitary wave minimizes E subject to $Q =$ const. Verify that Q comes from J by the general formula above.

Example 2: NLS.

$$iu_t - \Delta u - |u|^{p-1}u = 0, \quad x \in \mathbb{R}^n \tag{5.4}$$

where $u = u(r,t)$ vanishes in some sense as $r = |x| \to \infty$. A solitary wave solution is $u = \exp(i\omega t)\phi(x)$ where $\phi > 0$ satisfies the PDE

$$\Delta\phi + \omega\phi + |\phi|^{p-1}\phi = 0$$

for $\omega < 0$ and $1 < p < 1 + \frac{4}{n-2}$. The space is $X = H_r^1(\mathbb{R})$, the H^1 complex-valued functions that are radial. We can regard $\mathbb{C} = \mathbb{R}^2$. The group $G = \mathbb{R}$ acts on X by $u \to \exp(is)u$ (phase change). The skew operator is $J =$ multiplication by i. The invariants are

$$E(u) = \int_{\mathbb{R}^n} \left\{\frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1}\right\} dx, \quad Q(u) = \int_{\mathbb{R}^n} \frac{1}{2}|u|^2 dx.$$

Exercise: Verify that Q comes from J by the general formula above.

Return to general discussion.

In the abstract setting, we make the following assumptions.

Assumption 1: Local existence of solutions of the evolution equation, with invariants.

Assumption 2: Existence of solitary waves ϕ_ω depending on the parameter ω. They solve the stationary equation $E'(\phi_\omega) = \omega\, Q'(\phi_\omega)$ with ϕ_ω belonging to the appropriate domain and $T'(0)\phi_\omega \neq 0$. We have:

$$d(\omega) = E(\phi_\omega) - \omega Q(\phi_\omega) \text{ scalar}$$
$$0 = E'(\phi_\omega) - \omega\, Q'(\phi_\omega) \text{ vector}$$
$$H_\omega = E''(\phi_\omega) - \omega\, Q''(\phi_\omega) \text{ operator}$$

Exercise: $T'(0)\phi_\omega$ belongs to the kernel of H_ω.

Theorem 5.1. *Let $ker(H_\omega) = T'(0)\phi_\omega$ and let H_ω have exactly one negative eigenvalue and all its other spectrum positive and bounded away from 0. Assume $d''(\omega) \neq 0$. Then the solitary wave is stable (that is, nonlinearly orbitally stable) if and only if $d''(\omega) > 0$.*

This stability means that

$$\sup_{0\le t<\infty} \inf_{s\in\mathbb{R}} \|u(t) - T(s)\phi_\omega\|_X < \epsilon \quad \text{if} \quad \|u(0) - \phi_\omega\|_X < \delta.$$

5.2 Flavor of the proof

Idea: The solution remains on the manifold Q =constant, so we just need to check the linearized operator on $Q'(\phi)^\perp$, which is why we can allow a negative eigenvalue. We are assuming that H has mostly positive spectrum, which is a linear stability condition, and we just have to check the one remaining direction, which turns out to be controlled entirely by the sign of $d''(\omega)$. We can think of ϕ as providing the minimum of E subject to the constraint Q =constant, and regard ω as the Lagrange multiplier. Through most of the proof, ω is fixed so we often write $\phi = \phi_\omega$.

A key calculation: the derivative of $d(\omega)$ is

$$d'(\omega) = \left\langle E'(\phi) - \omega Q'(\phi), \frac{d\phi}{d\omega} \right\rangle - Q(\phi) = -Q(\phi), \quad d''(\omega) = -\left\langle Q'(\phi), \frac{d\phi}{d\omega} \right\rangle.$$

But $0 = E'(\phi) - \omega\, Q'(\phi)$ implies $H(d\phi/d\omega) = Q'(\phi)$, so that

$$d''(\omega) = -\left\langle H\frac{d\phi}{d\omega}, \frac{d\phi}{d\omega} \right\rangle. \tag{5.5}$$

There are two geometric structures in X: the 2D surface $(s, \omega) \to T(s)\phi_\omega$ and the level hypersurfaces of Q. For fixed ω, consider the tubular neighborhood

$$U_\epsilon = \{u \in X : \inf_{s\in\mathbb{R}} \|u - T(s)\phi\| < \epsilon\}.$$

Stability:

The assumption $d''(\omega) > 0$ means that $d\phi/d\omega$ is a negative vector for H. If $y \in Q'(\phi)^{\perp}$, then $0 = \langle Q'(\phi), y\rangle = \langle H(d\phi/d\omega), y\rangle$. Therefore y points outside the negative cone of H. (Why?) That is,

$$\langle Hy, y\rangle > 0, \quad \forall y \ \in Q'(\phi)^{\perp}.$$

This is our stability condition in linearized form. In order to obtain nonlinear stability, we first show that this linear stability condition provides a lower bound for $E(u) - E(\phi)$.

Lemma 5.2 (Main Lemma)**.** *Fix ω and let $\phi = \phi_\omega$. Let $d''(\omega) > 0$. The $\exists\ C > 0, \epsilon > 0$ such that*

$$E(u) - E(\phi) \geq C \inf_{s\in\mathbb{R}} \|u - T(s)\phi\|^2, \quad \forall u \in U_\epsilon,\ Q(u) = Q(\phi).$$

We omit the proof.

Proof of stability. By contradiction. If unstable, $\exists\delta_0 > 0$ and a sequence $u_n(0)$ such that

$$\|u_n(0) - \phi\| \to 0 \quad \text{but} \quad \sup_{0\leq t<\infty} \inf_s \|u_n(t) - T(s)\phi\| \geq \delta_0. \tag{5.6}$$

Choose $\|u_n(0) - \phi\| < \epsilon/2$. By Assumption 1, we can pick the first time t_n for which

$$\inf_s \|u_n(t_n) - T(s)\phi\| = min(\delta_0, \epsilon/2).$$

Then by the invariance of E and Q,

$$E(u_n(t_n)) = E(u_n(0)) \to E(\phi), \quad Q(u_n(t_n)) = Q(u_n(0)) \to Q(\phi).$$

Now choose v_n such that $Q(v_n) = Q(\phi)$ and $\|v_n - u_n(t_n)\| \to 0$.
Exercise: How do we do this?
Then $E(v_n) - E(u_n(t_n)) \to 0$, so that $E(v_n) \to E(\phi)$. Applying the Main Lemma to v_n, we see that $\inf_s \|v_n - T(s)\phi\| \to 0$, so that $\inf_s \|u_n(t_n) - T(s)\phi\| \to 0$, which contradicts the instability presumption. □

Instability:

Lemma 5.3. *Let $d''(\omega) < 0$. Then E restricted to Q =const is NOT locally minimized at ϕ_ω.*

Proof. We'll use the dependence on ω. Let χ be the negative eigenvector of H. This is the unstable direction. From $d''(\omega) < 0$ we have $\langle Q'(\phi_\omega), d\phi_\omega/d\omega\rangle > 0$. Consider the curve $\Omega \to \phi_\Omega$ in X for Ω near ω. By the IFT, we locally find $\Omega(s)$ that solves the equation $Q(\phi_{\Omega(s)} + s\chi) = Q(\phi_\omega)$. Now if we Taylor expand $E(\phi_\Omega + s\chi) - \Omega Q(\phi_\Omega + s\chi)$ around ϕ_Ω and then put $\Omega = \Omega(s)$, we can get $E(\phi_{\Omega(s)} + s\chi)) < E(\phi_\omega)$ for small s. □

Proof of instability (sketch). In fact, we can find a vector y for which $\langle Hy, y\rangle < 0$ and $\langle Q'(\phi), y\rangle = 0$. Choose a curve $\psi(\cdot)$ on the hypersurface Q =const such that $\dot\psi(0) = y,\ \psi(0) = \phi$. Then $E(\psi(s))$ has a strict local maximum at $s = 0$ since

$$E(\psi)^{\cdot} = \langle E'(\phi) - \omega Q'(\phi), y\rangle = 0, \quad E(\psi)^{\cdot\cdot} = \langle Hy, y\rangle < 0$$

there. We choose initial data $u(0) = \psi(s)$ for some small s and solve the PDE locally. For each $v \in X$, choose the point $T(s(v)u)$ to be the point on the orbit $T(s)v$ which is closest to ϕ. Then define

$$A(u) = -\langle J^{-1}y, T(s(u))u\rangle.$$

Then we differentiate and eventually prove that

$$\frac{d}{dt}A(u(t)) = \langle A'(u(t)), \frac{du}{dt}\rangle = -\langle E'(u), JA'(u)\rangle > \epsilon_0$$

for some positive constant ϵ_0. The inequality is due to a Taylor expansion that begins with $-\langle E'(u), JA'(u)\rangle = -\langle Hy, JJ^{-1}y\rangle + \ldots$ Therefore $A(u(t))$ grows at least linearly in t. But $|A(u(t))| \leq \|u(t)\|$ so that $\|u(t)\|$ also grows. So we have instability. □

5.3 Back to the examples

For the **peakon** example with $k > 0$, the group $G = \mathbb{R}$ acts on $X = H^1(\mathbb{R})$ by translation $u(x) \to u(x-s)$. The skew operator is $J = \frac{1}{2}\partial(1-\partial^2)^{-1}$. The invariants are

$$Q(u) = -\int_{\mathbb{R}}(u^2 + u_x^2)dx, \quad E(u) = -\int_{\mathbb{R}}(u^3 + uu_x^2 + 2ku^2)dx.$$

Exercise: Using the ODE for $\phi = \phi_\omega$ with $\omega > 2k$, calculate

$$d'(\omega) = -Q(\phi) = 4\int_{2k}^{\omega}\frac{(\omega - y)(y-k)}{\sqrt{y(y-2k)}}dy.$$

Exercise: Also calculate

$$H = -\partial\{2(\omega - \phi)\partial\} - 6\phi + 2\phi_{xx} + 2(\omega - 2k).$$

Thus $d''(\omega) > 0$. Now $H(\phi_x) = 0$ and ϕ_x has exactly one zero. So the Sturm-Liouville operator H has one negative eigenvalue, one zero eigenvalue, and the rest is $\geq 2\omega - 2k > 0$. Therefore ϕ is stable.

For the **NLS** example, splitting into real and imaginary parts, the linearized Hamiltonian is

$$H = \begin{pmatrix} -\Delta - p\phi^{p-1} - \omega & 0 \\ 0 & -\Delta - \phi^{p-1} - \omega \end{pmatrix}, \tag{5.7}$$

which again has the required spectral properties. Furthermore, we have $-\Delta\phi_\omega - \omega\phi_\omega - |\phi_\omega|^p = 0$ with $\omega < 0$ and $1 < p < 1 + \frac{4}{n-2}$. Changing scale and letting $\lambda = \sqrt{-\omega}$, we have $\phi_\omega(x) = \lambda^{2/(p-1)}\zeta(\lambda x)$. Thus

$$d'(\omega) = -\frac{1}{2}\int|\phi_\omega|^2dx = -(\text{pos const})\ \lambda^{-n+4/(p-1)}$$

whence

$$d''(\omega) = (\text{pos const})\ (-n + \frac{4}{p-1})(-\omega)^{\frac{1}{2}(-n+\frac{4}{p-1})-1}.$$

Therefore we have stability if $1 < p < 1 + \frac{4}{n}$ and instability if $1 + \frac{4}{n} < p < 1 + \frac{4}{n-2}$.