AGNES at Brown 10/28/2012 "Algebraic & tropical curves"

Let R be a DVR, $K = \mathrm{Frac}\,R$, and $k = R/m$ algebraically closed.

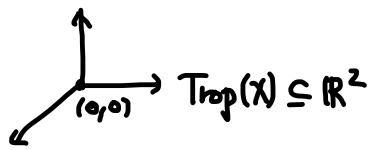
<u>Example</u>: $R = \mathbb{C}[[t]]$, $K = \mathbb{C}((t))$, $k = \mathbb{C}$.

Let X be a subvariety of $(K^*)^n$.

The (embedded) tropicalization is

$$\mathrm{Trop}(X) = \{(\mathrm{val}(x_1), \dots, \mathrm{val}(x_n)) : (x_1, \dots, x_n) \in X(K') \text{ for } K'/K \text{ any valued field extension}\} \subseteq \mathbb{R}^n.$$

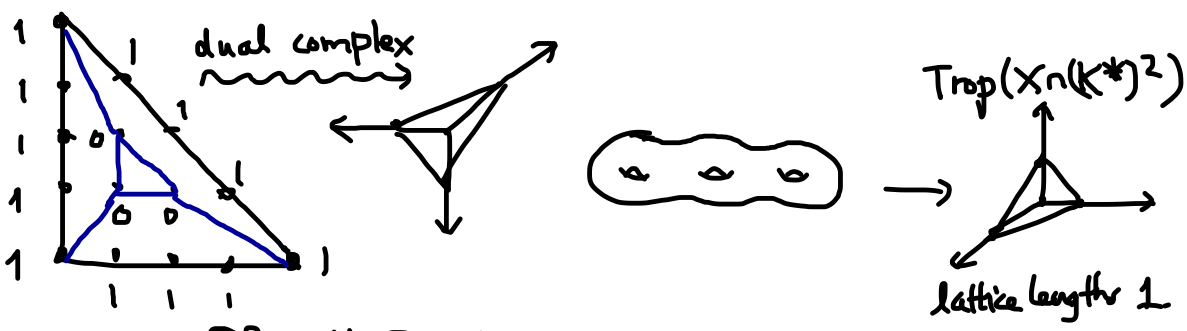
<u>Example</u>: $X = V(x+y+1) \subseteq (K^*)^2$.



$\mathrm{Trop}(X)$ is a polyhedral complex of dimension $\dim(X)$, balanced, ...

It's easy to compute tropical hypersurfaces:

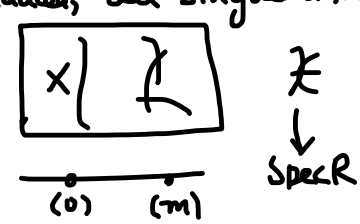
<u>Ex.</u> $X = V(x^2yz + xy^2z + xyz^2 + t(x^4 + x^3y + x^2y^2 + xy^3 + y^4 + y^3z + \cdots + z^4)) \subseteq \mathbb{P}^2_K$.



In the general case: use Gfan (A. Jensen).

<u>Towards (abstract) tropicalizations</u>. Let X smooth, proper curve$/K$. Let $\mathcal{X}$ proper, flat scheme$/R$, regular surface, such that

- generic fiber $\cong X$
- special fiber $\mathcal{X}_k$ is a semistable curve over k: reduced, all singularities are ordinary double points, (no self-nodes)



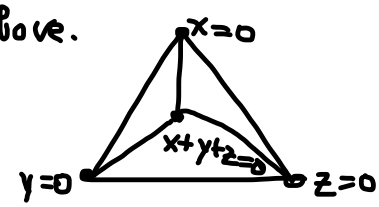
$\mathcal{X}$ is called a <u>semistable model</u> for X.

We associate to X the <u>dual graph</u> G of $\mathcal{X}_k$

(vertices $\longleftrightarrow$ components, edges $\longleftrightarrow$ nodes)

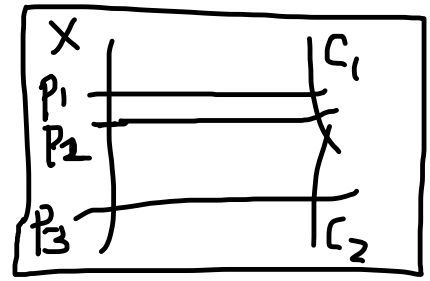
<u>Example</u> $\mathcal{X} = V(x^2yz + xy^2z + xyz^2 + t(\cdots)) \subset \mathbb{P}^2_R$ as above.

$\mathcal{X}_k = V(xyz(x+y+z)) \subset \mathbb{P}^2_{\mathbb{C}}$.

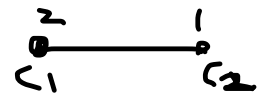


<u>In fact</u>: $\exists$ toric variety Y_Δ and $X \hookrightarrow Y_\Delta$ such that $\mathrm{Trop}\,X$ contains a (homeomorphic) copy of G. [Thuillier, Payne]

In fact we may define a map $\mathrm{Div}X \xrightarrow{\rho} \mathrm{Div}\,G$.
We'll show $\mathrm{rk}_X(D) \le \mathrm{rk}_G(\rho(D))$ [Baker]



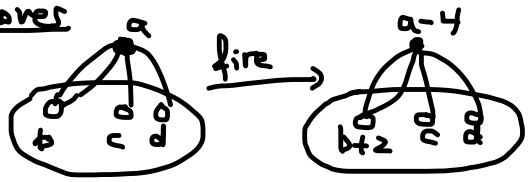
$\rho(p_1+p_2+p_3) = 2C_1 + C_2$

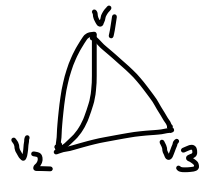


<u>Divisors on graphs</u> (following Baker-Norine)

Let $\mathrm{Div}\,G$ = formal $\mathbb{Z}$-sums of vertices.

Say $D \sim D'$ are <u>linearly equivalent</u> if D' obtained from D via <u>chip-firing moves</u>





$$\begin{bmatrix} -3 & 2 & 1 \\ 2 & -3 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

<u>Complete linear system</u> of $D \in \mathrm{Div}\,G$ is $|D| = \{E \in \mathrm{Div}\,G : E \ge 0,\ D \sim E\}$

<u>Rank</u> of D is $\mathrm{rk}_G(D) = \max\{k \in \mathbb{Z} : \forall E \text{ effective of degree } k,\ |D-E| \ne \emptyset\}$

<u>Def</u> Specialization map: $\rho : \mathrm{Div}\,\mathcal{X} \longrightarrow \mathrm{Div}\,G$

$$\mathcal{D} \longmapsto \sum_1^n (C_i \cdot \mathcal{D}) \cdot C_i$$

where $\{C_1, \dots, C_n\}$ are the irreducible components of $\mathcal{X}_k$.

Extend to $\rho : \mathrm{Div}\,X \longrightarrow \mathrm{Div}\,G$ via $\rho(x) = \rho(\overline{\{x\}})$.

<u>Specialization theorem</u> (Baker) $\forall D \in \mathrm{Div}\,X$,

$$\mathrm{rk}_G(\rho(D)) \ge \mathrm{rk}_X(D).$$

<u>Generalizations</u>: Amini-Caporaso, Amini-Baker, Cartwright, ...

<u>Application</u> • Tropical proof of Brill-Noether theorem (Cools-Draisma-Payne-Robeva)

• existence theorem for graphs: if $g-(r+1)(g-d+r) \ge 0$ then $\forall \Gamma$ graph of genus g, $\exists\, g^r_d$ on Γ.

<u>Abstract tropical curves</u>. An abstract tropical curve is (G, w, ℓ) where

- G graph (connected)
- $w : V(G) \longrightarrow \mathbb{Z}_{\ge 0}$
- $\ell : E(G) \longrightarrow \mathbb{R}_{>0}$

s.t. every weight 0 vertex has valence ≥ 3, every weight 1 vertex has valence ≥ 1.

<u>Genus</u> of (G,w,ℓ) is $b_1(G)+\sum w(v) = \sum w(v)+|E|-|V|+1$.

Let K be a nonarchimedean field, $\mathrm{val}: K \to \mathbb{R}\cup\{\infty\}$, R valuation ring, $k=R/m$ algebraically closed. A <u>semistable model</u> for X/K is a proper, flat scheme $\mathfrak{X}/R$ with generic fiber $\cong X$ and $\mathfrak{X}_k$ semistable$/k$.

If $x\in\mathfrak{X}$ node in $\mathfrak{X}_k$, then $\hat{\mathcal{O}}_{\mathfrak{X},x} = \dfrac{\hat{R}[[u,v]]}{(uv-a)}$ for $a\in R$.

Equip edge corresponding to x with length $\mathrm{val}(a)$ "thickness"

<u>Ex</u> If $R=\mathbb{C}[[t]]$ and $\mathfrak{X}$ regular, then $\hat{\mathcal{O}}_{\mathfrak{X},x} \cong \dfrac{R[[u,v]]}{(uv-t)} \Rightarrow$ all edge lengths $=1$.

Thus we have defined a map

$$\mathcal{M}_g(K) \xrightarrow{\text{trop}} M_g^{\text{trop}} = \{\text{genus } g \text{ tropical curves}\}$$

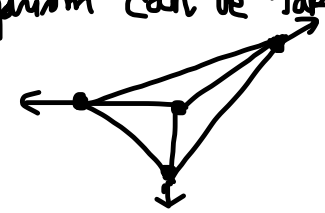
$$X/K \longmapsto \text{(stabilized) dual graph } G \text{ of } \mathfrak{X}_k \text{ for } \mathfrak{X} \text{ a ss model,}$$

$$w(C_i) := g(C_i)$$

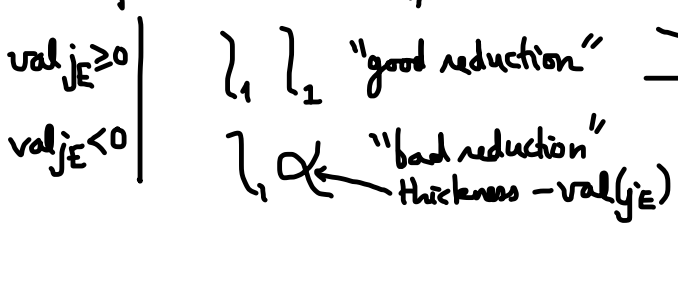
$$\ell(\text{node}) = \text{thickness of } x.$$

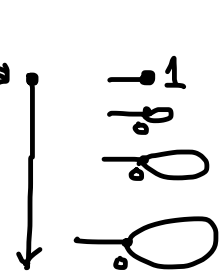
<u>Remark</u>: This is intrinsic: $\exists$ canonical deformation retract $X^{an} \to \mathrm{Trop}(X)$

<u>Thm</u> (Baker-Payne-Rabinoff) Payne's homeomorphism can be taken to be an isometry (with respect to <u>lattice length</u>).



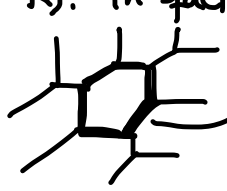
<u>Example</u> $(K=\bar{K})$ $\mathcal{M}_{1,1}(K) \longrightarrow M_{1,1}^{\text{trop}}$





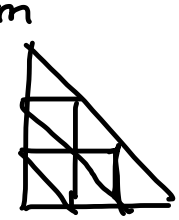
$M_{1,1}^{\text{trop}}$ is a ray.

Let E/K elliptic curve with $\mathrm{val}(j_E)<0$. Then $\exists$ map $E \xhookrightarrow{\varphi} \mathbb{P}^2_K$ such that $\mathrm{Trop}(E\cap(K^*)^2)$ contains an isometric copy of the cycle of length $-\mathrm{val}(j_E)$ (B-P-R). In fact we may choose φ s.t. $\mathrm{Trop}\,\varphi(E)$ has form



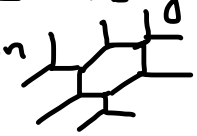
hexagon has lattice length $-\mathrm{val}(j_E)$

$\longleftarrow$ dual to



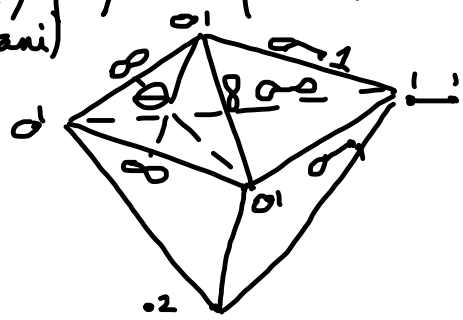
<u>Thm/Algorithm</u> (C–Sturmfels)

$j \in K$ with $\mathrm{val}(j) < 0$ $\rightsquigarrow$ ten coefficients $c_{300}, \dots, c_{003}$ s.t. $c_{300}X^3 + \dots + c_{003}Z^3$ has j-invariant j and tropicalization



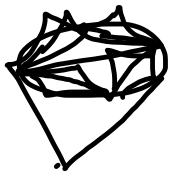
<u>What is M_g^{trop}?</u> (Mikhalkin–Zharkov, Brannetti–Melo–Viviani, Abramovich–Caporaso–Payne) It is a "stacky polyhedral fan" (B–M–V) with combinatorics as described in (C, Maggioli–Pagani)

<u>Ex</u> $g=2$



<u>Thm</u> (C–Melo–Viviani)

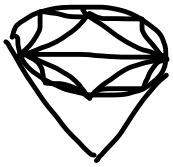
We have a commutative diagram of "ideal stacky polyhedral fans"

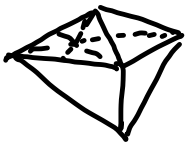


"tropical Teichmüller Space" outer space of rank g

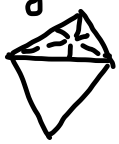
$$\begin{tikzcd} \overline{X}_g \arrow[r] \arrow[d, "/\!/ \mathrm{Out}\,F_g"] & \overline{\Omega}_g \arrow[d, "/\!/ GL_g(\mathbb{Z})"] \\ M_g^{trop} \arrow[r, "\text{Torelli}", "(B\text{-}M\text{-}V,\ C)"'] & A_g^{trop} \end{tikzcd}$$

"tropical Siegel space"





"tropical moduli space"



"moduli space of p.p. tropical abelian varieties"