AGNES 10/12

Singular support of coherent sheaves. (on l.c.i.)

(present a tool: no time for any...)

① Let X be an affine complete intersection / $\mathbb{C}$ char $k = 0$.

$[\mathbb{Z}(f_1,\dots,f_n)]$ $X = \operatorname{Spec} A$, $A = R/(f_1,\dots,f_n)$. [R = regular algebra]

$\cap$

$S = \operatorname{Spec} R$ - smooth.

Rm If (f_i) is not a reg. sequence, X is a dg-scheme.

Key observation: Gulliksen ~74: For any A-module F (or any $F \in D(X)$), there are $\xi_1,\dots,\xi_n \in \operatorname{Ext}^2(F,F)$. [derived cat].

Example: $A = R/(f)$. Consider $i: X \to S$. [Easy calculation] given a q.coh. sheaf F on X,

$$L_k i^* i_* F = \begin{cases} F, & k = 0,1 \\ 0, & \text{otherwise}. \end{cases}$$

In $D(X)$

$$F[1] \to Li^* i_* F \to F \to \text{ gives}$$

$$\xi : F \to F[2]$$

$$\xi \in \operatorname{Ext}^2(F,F).$$

Used by Eisenbud, Avramov - Buchweitz and others.

This gives morphism

$$A[\xi_1,\dots,\xi_n] \to \operatorname{Ext}^\bullet(F,F) = \bigoplus_k \operatorname{Ext}^k(F,F)$$

$(\deg \xi_i = 2)$

[ξ_i's commute]

Key Theorem (Gulliksen) Suppose $F \in D^b_{coh}(X)$. Then $\mathrm{Ext}^\bullet(F,F)$ is finitely generated module over $A[\xi_1,\dots,\xi_n]$.

[Nice exercise (in Koszul's kind of constructions).]

Key Definition Given $F \in D^b_{coh}(X)$, set
$$\mathrm{Sing\,Supp}(F) = \mathrm{supp}_{A[\xi_1,\dots,\xi_n]} \mathrm{Ext}^\bullet(F,F) \subset X \times \mathbb{C}^n$$

Due to Avramov-Buchweitz (if F is f. length), Krause-Iyengar-Benson in general and more.

Remark In fact: $A[\xi_1,\dots,\xi_n] \to HH^\bullet(X) \to \mathrm{Ext}^\bullet(F,F)$.

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② As a complex, T^*X (or $\mathbb{L}X$) is
$$\mathcal{O}_X^n \xrightarrow[(df_1,\dots,df_n)]{d} (T^*S)|_X$$

[naive] $H^0(T^*X) = \mathrm{coker}(d)$

$H^{-1}(T^*X) = \ker(d) = \{(x, a_1,\dots,a_n) \mid \sum a_i df_i(x) = 0\} \subset X \times \mathbb{C}^n$

Claim For any $F \in D^b_{coh}(X)$, $\mathrm{Sing\,Supp}(F) \subset H^{-1}(T^*X)$.

Sing Supp is local in Zariski top. and independent of presentation $A = R/(f_1,\dots,f_n)$, so makes sense for any l.c.i. X (could be a stack, too).

Analogy:

Coherent D-module M on smooth $X \to$ conical $\mathrm{Sing\,Supp}(M) \subset T^*X$

$\mathrm{Sing\,Supp}(M) \cap (0 \text{ section}) = \mathrm{Supp}\, M$,

$\mathrm{Sing\,Supp}(M) - (0 \text{ section})$ tells that M is not a local system

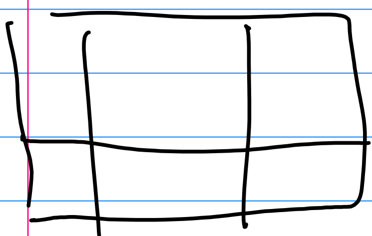
Coherent sheaf F on l.c.i. $X \longrightarrow \mathrm{SingSupp}(F) \subset \mathcal{H}^{-1}(T^*X)$,

$\mathrm{SingSupp}(F) \cap (\{0\}\text{-section}) = \mathrm{Supp}\, F$

(infinite Tor-dimension) $\left\{ \begin{array}{l} \mathrm{SingSupp}(F) - \{0\}\text{-section} = \\ \text{tells that } F \text{ is not perfect.} \end{array} \right.$

Example $X = \mathrm{Spec}\, R/f$ = singular hypersurface.

$$\mathcal{H}^{-1}T^*X = (\underbrace{X\times\{0\}}_{\text{zero section}}) \cup (\mathrm{Sing}\,X)\times\mathbb{C} \subset X\times\mathbb{C}$$



$$\mathrm{SingSupp}(F) = (\mathrm{Supp}\,F\times\{0\}) \cup \left(\begin{pmatrix}\text{non-perfect}\\ \text{locus of } F\end{pmatrix}\times\mathbb{C}\right)$$

Remark $\mathrm{SingSupp}(F) - (0\text{-section})$ is defined on category of singularities of X; Orlov's Thms give an equivalent computation.

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③ Ind-coherent sheaves [Can't deal with coherent sheaves all the time]

$$X \longmapsto \mathrm{Ind\,Coh}(X) \supset D(X)$$

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Defs:

① Krause: $\mathrm{Ind\,Coh}(X) = \{\text{complexes of inj. modules}\}/\text{homotopy}$

② Positselski: $\mathrm{Ind\,Coh}(X)$ = coderived category

③ $\mathrm{Ind\,Coh}(X)$ = Ind completion of DG-category $D^b_{coh}(X)$

$\mathrm{Ind\,Coh}(X)$ is compactly generated by $D^b_{coh}(X)$

$D(X)$ is compactly generated by Perfect complexes

Standard game. Given $\mathcal{F} \in \mathrm{IndCoh}(X)$, consider $G \in D^b_{coh}(X) \subset \mathrm{IndCoh}(X)$; ~~and~~ cohomological operators

$$\underset{\circlearrowleft}{\xi_i} \text{ act on } \underset{\mathrm{IndCoh}(X)}{\mathrm{Hom}}(G, F).$$

We say that $\mathrm{SingSupp}(\mathcal{F}) = \overline{\bigcup} \mathrm{supp}_{A[\xi_1,..,\xi_n]} \mathcal{H}om(G,F)$. It is a conical subset of $H^{-1}T^*X$.

(4) Why?

Classical: $X \rightsquigarrow D(X)$; $f: X_1 \to X_2$ gives f_*, f^*.

Alternative: $X \rightsquigarrow \mathrm{IndCoh}(X)$; $f: X_1 \to X_2$ gives $f_*, f^!$

Variations: For any conical $Y \subset H^{-1}T^*X$,

$X \rightsquigarrow \mathrm{IndCoh}_Y(X) = \{\mathcal{F} \in \mathrm{IndCoh}(X) \mid \mathrm{SingSupp}(\mathcal{F}) \subset Y\}$ functorial in natural way.

Appears in FM of singular varieties.

Motivating example

(Conjectural) Langlands transform: kind of Laumon-Rothstein

$$\text{D-mod on Bun} \longleftrightarrow \text{O-mod on LocSys.}$$

Inconsistent. singular; l.c.i

Fact (A., Gaitsgory) Becomes consistent if RHS is $\mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LocSys})$ for chosen Y = nilp cone.