Artin fans

AMS special session on Combinatorics and Algebraic Geometry

Dan Abramovich

Brown University

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Heros:

- Martin Olsson
- Jonathan Wise
- Qile Chen, Steffen Marcus,
- Mark Gross, Bernd Siebert
- Martin Ulirsch

Superabundance

Mikhalkin-Speyer: there is a tropical cubic curve C of genus 1 in TP^3 which does not lift to an algebraic curve (Speyer, *Tropical Geometry*, Berkeley thesis 2005, Figure 5.1).

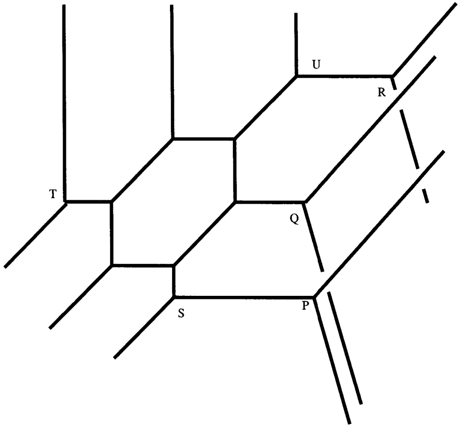

Figure 5.1: A Genus 1 Zero Tension Curve which is not Tropical

Superabundance (continued)

I want to understand this phenomenon.

Principles:

- Tropical curves in TP^3 encode degenerations of curves in $\mathbb{P}^3$
- They encode in detail the manner in which they degenerate
- They encode **logarithmic stable maps** in $\mathbb{P}^3$.
- But logarithmic stable maps are **obstructed**.

Question

Is there a world in which they are not obstructed?

Logarithmic structures

Definition

A *pre logarithmic structure* is

$$X = (\underline{X}, M \overset{\alpha}{\to} \mathcal{O}_{\underline{X}}) \qquad \text{or just} \quad (\underline{X}, M)$$

such that

- $\underline{X}$ is a scheme - the *underlying scheme*
- M is a sheaf of monoids on X, and
- α is a monoid homomorphism, where the monoid structure on $\mathcal{O}_{\underline{X}}$ is the multiplicative structure.

Definition

It is a *logarithmic structure* if $\alpha : \alpha^{-1}\mathcal{O}^*_{\underline{X}} \to \mathcal{O}^*_{\underline{X}}$ is an isomorphism.

Examples

Examples

- $(\underline{X}, \mathcal{O}^*_{\underline{X}} \hookrightarrow \mathcal{O}_{\underline{X}})$, the trivial *logarithmic* structure.
- Let $\underline{X}, D \subset \underline{X}$ be a variety with a divisor. We define $M_D \hookrightarrow \mathcal{O}_{\underline{X}}$:

$$M_D(U) = \left\{ f \in \mathcal{O}_{\underline{X}}(U) \;\middle|\; f_{U \smallsetminus D} \in \mathcal{O}^\times_{\underline{X}}(U \smallsetminus D) \right\}.$$

- Let k be a field,

$$\begin{array}{ccc} \mathbb{N} \oplus k^\times & \to & k \\ (n, z) & \mapsto & z \cdot 0^n \end{array}$$

defined by sending $0 \mapsto 1$ and $n \mapsto 0$ otherwise.

The magic of logarithmic geomery

- Any toric variety X is logarithmically smooth

$$T_X \simeq \mathcal{O}_{\underline{X}}^{\dim \underline{X}}.$$

- A nodal curve is logarithmically smooth over a logarithmic point.

Here be monsters!

Logarithmic obstructions to deforming a logarithmic map $C \to \mathbb{P}^3$ lie in

$$H^1(\underline{C}, \mathcal{O}_{\underline{C}}^3).$$

These can be nonzero on a broken cubic curve!

Artin fans

Olsson:

$$\{\text{Logarithmic structures } X \text{ on } \underline{X}\} \quad \longleftrightarrow \quad \{\underline{X} \to \underline{\text{Log}}\}.$$

The stack Log *is huge and does not specify combinatorial data.*

Proposition (Wise; ℵ, Chen, Marcus)

There is an initial factorization $X \to \mathcal{A}_X \to \text{Log}$ *such that* $\mathcal{A}_X \to \text{Log}$ *is étale, representable, strict.*

The stack $\mathcal{A}_X$ is small, totally combinatorial.

$\mathbb{P}^3$ and $\mathcal{A}_{\mathbb{P}^3}$

$$\mathbb{P}^3 = (\mathbb{A}^4 \smallsetminus \{0\})/\mathbb{G}_m.$$

So

$$\{C \to \mathbb{P}^3\} \leftrightarrow \{(\mathcal{L}, s_0, \ldots, s_3) | s_i \text{ do not vanish together}\}.$$

Now

$$\mathcal{A}_{\mathbb{P}^3} = (\mathbb{A}^4 \smallsetminus \{0\})/\mathbb{G}_m^4.$$

So

$$\{C \to \mathcal{A}_{\mathbb{P}^3}\} \leftrightarrow \{((\mathcal{L}_0, s_0), \ldots, (\mathcal{L}_3, s_3)) | s_i \text{ do not vanish together}\}.$$

The monsters evaporate!

$$T_{\mathbb{P}^3} = \mathcal{O}^3, \text{ but } T_{\mathcal{A}_{\mathbb{P}^3}} = 0.$$

Logarithmic obstructions to deforming a logarithmic map $C \to \mathcal{A}_{\mathbb{P}^3}$ lie in

$$H^1(\underline{C}, 0).$$

The obstructions are gone!

Sample theorem

Theorem (ℵ-Wise)

If $Y \to X$ is a toric modification, then

> *Logarithmic Gromov–Witten invariants of X coincide with those of Y.*

Reason: $\mathfrak{M}(\mathcal{A}_Y) \to \mathfrak{M}(\mathcal{A}_X)$ is birational. So $\overline{\mathcal{M}}(Y) \to \overline{\mathcal{M}}(X)$ is virtually birational.

Tropicalization

Things are connected in Martin Ulirsch's fundamental commutative diagram:

$$\begin{array}{ccccc}
X^{\beth} & \longrightarrow & \mathcal{A}_X^{\beth} & \longrightarrow & \Sigma_X \\
\rho_X \downarrow\downarrow r_X & & \rho_{\mathcal{A}} \downarrow\downarrow r_{\mathcal{A}} & & \rho_{\Sigma} \downarrow\downarrow r_{\Sigma} \\
X & \longrightarrow & \mathcal{A}_X & \longrightarrow & F_X
\end{array}$$

- $F_{\mathbb{P}^3} = \mathbb{P}^3_{\mathbb{F}_1}$ $\qquad$ $\Sigma_{\mathbb{P}^3} = \overline{TP}^3$.
- $X^{\beth}$ - Berkovich analytic formal fiber
- $\mathbb{P}^3$ and $\mathcal{A}_{\mathbb{P}^3}$ share their tropicalization $\overline{TP}^3$.
- $\mathcal{A}_X^{\beth} \to \Sigma_X$ **is a homeomorphism**.