LOGARITHMIC PLANS

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All log schemes here should probably be fine and saturated.

1. Log Points and their Maps

1.1. The stack of standard log points.

Definition 1.2. A standard log points is a point with log structure such that the characteristic is \( \mathbb{N} \). Define a family of standard log points over a log scheme \( S \) as follows: it is a log scheme \( S' \) with a morphism \( S' \to S \) which is an isomorphism on underlying schemes, and such that that \( \overline{M}_S = \overline{M}_S[m_x] \). Arrows of such families are defined by cartesian diagrams as usual.

Problem 1.3. Show the following:

This defines a stack over \( \log\mathbf{Sch} \).

This stack is isomorphic to \( \mathcal{B} \mathbb{G}_m \), with the trivial log structure. The universal family is \( \mathcal{B} \mathbb{G}_m \) with the log structure inherited from the embedding \( \mathcal{B} \mathbb{G}_m \subset [\mathbb{A}^1/\mathbb{G}_m] \).

Now let \( X \) be a log scheme. Define another category \( \wedge X \) over \( \log\mathbf{Sch} \) whose objects over \( S \) are \((S' \to S, S' \to X)\) where \( S' \to S \) is a family of standard log points and \( S' \to X \) a morphism, and arrows by cartesian diagrams.

Problem 1.4. This stack \( \wedge X \) is a representable countable union of log schemes projective over \( X \).

Problem 1.5. Give a concrete description of \( \wedge X \). Show that when \( X \) is log smooth this thing is also log smooth.

1.6. The stack of standard log nodes.

Definition 1.7. Define a family of standard log nodes over a log scheme \( S \) as follows: it is a log scheme \( S' \) with a homomorphism \( \phi : \mathbb{N} \to \overline{M}_S \) and a morphism \( \pi : S' \to S \) which is an isomorphism on underlying schemes, and such that that \( \overline{M}'_S = \overline{M}_S[m_x, m_y]/(m_x + m_y = \phi(1)) \). Arrows defined by cartesian diagrams. This defines a category \( \mathbf{Nodes} \) over \( \log\mathbf{Sch} \).

Problem 1.8. Figure out the right definition, whether or not this is it. Show it is an algebraic log stack. Describe the stack \( \mathbf{Nodes} \) in a reasonable way.

Date: today.
Now let $X$ be a log scheme. Define another stack $\wedge^{\log} X$ over $\LogSch$ whose objects over $S$ are $(S' \to S, S' \to X)$ where $S' \to S$ is a family of standard log nodes and $S' \to X$ a morephism, and arrows by cartesian diagrams.

**Problem 1.9.** Give a concrete description of $\wedge^{\log} X$.

## 2. Log smooth GW theory

### 2.1. Evaluation maps.

Consider a log smooth scheme $X$. We have a stack $K_\Gamma(X)$ of log stable maps to $X$ with numerical invariants $\Gamma$ (including genus, types of markings, class etc.).

**Problem 2.2.** Show that the restriction of the universal map $C^{univ} \to X$ to the $i$-th marking gives a morphism $e_i : K_\Gamma(X) \to \wedge X$.

### 2.3. Obstruction theory.

**Problem 2.4.** Show that the log cotangent complexes of $X$ and of $C/K$ are their sheaves of log differentials, so locally free, and there is a morphism $f^*\Omega_X \to \Omega_{C/S}$.

**Problem 2.5.** Show that the standard formalism provides a perfect log obstruction theory $E \to \mathbb{L}_{K_\Gamma(X)}$.

**Problem 2.6.** We have an isomorphism $\mathbb{L}_{K_\Gamma(X)} \simeq L(L_{K_\Gamma(X)}/\Log)$, where $K_\Gamma(X)$ is the stack underlying $K_\Gamma(X)$.

### 2.7. Virtual fundamental class.

**Problem 2.8.** Show that the image of $K_\Gamma(X)$ in $\Log$ lies in an open sub-stack $\Log^0$, of finite type over $\mathbb{C}$, of pure dimension 0, satisfying Kresch’s stratification conditions [1], so a fundamental class $[\Log^0]$ exists.

**Problem 2.9.** Show that the morphism $K_\Gamma(X) \to \Log^0$ and the log obstruction theory $E \to \mathbb{L}_{K_\Gamma(X)}$ satisfy Manolescu’s requirements [2], so a refined pull-back $f_E^! : A_*(\Log^0) \to A_*(K_\Gamma(X))$ exists.

**Problem 2.10.** Investigate the resulting virtual fundamental class $[K_\Gamma(X)]^{vir} = f_E^!(\Log^0)$ and its properties.

### 2.11. Invariants.

**Problem 2.12.** Define log GW invariants as usual by

$$\langle \gamma_1 \cdots \gamma_n \rangle := \int_{[K_\Gamma(X)]^{vir}} \prod e_i^* \gamma_i.$$

Define analogous descendant invariants. Investigate their basic properties.
2.13. **Localization.**

**Problem 2.14.** Extend the standard theory of virtual localization in the log smooth context.

3. **Log nodal theory**

**Definition 3.1.** Define a log prenodal curve to be a subcurve of a log smooth curve.

Define a log presemistable variety to be a subvariety which is a union of components of a semistable (or maybe log smooth) variety.

**Problem 3.2.** Define log stable maps of log prenodal curves into log schemes.

**Problem 3.3.** Compare log stable maps of log prenodal curves, maybe with log presemistable targets, with log stable maps of log smooth curves into log smooth targets.

**Problem 3.4.** Define prenodal evaluation maps.

**Problem 3.5.** Investigate when stable log prenodal maps admit a perfect log obstruction theory.

**Problem 3.6.** In such situations, define prenodal GW invariants. Compare with the log smooth GW invariants.

**Problem 3.7.** Define Gross-Hacking-Keel invariants of a normal crossings variety.

4. **Degeneration formula**

**Problem 4.1.** Investigate log degeneration formulas in the higher rank case (the case of a smooth divisor case this should be in Qile’s thesis). Use wither the log prenodal formalism or through a connection with the log smooth theory.

5. **Other aspects**

**Problem 5.1.** In the smooth divisor case, compare the log GW theory with Kim’s theory.

**Problem 5.2.** Prove an existence result for general log maps (source not necessarily a point or curve).

**References**