Very basic exercises on logarithmic structures

- Terrible, Horrible, No Good, Very Bad log structures
  (1) Consider the monoid P generated by 2 and 3 in $\mathbb{N}$, and let $(\operatorname{Spec} k, M)$ be the log structure associated to the standard map "$0^n : P \to k$". What's wrong with this log structure? Can it be improved?
  (2) Consider the same monoid, let X be the cuspidal curve $y^2 = x^3$. Consider $P \to \mathcal{O}_X$ defined by $2 \mapsto x$ and $3 \mapsto y$. Want do you think about the associated log structure? Can this one be improved?
  (3) Compare the above log structure on X with the ones defined by $\mathbb{N} \to \mathcal{O}_X$ defined by $1 \mapsto x$ or $1 \mapsto y$. (You may want to contemplate the relationship of normalization with log structures, at least in this case).
  (4) Consider the monoid $M = k^* \sqcup \mathbb{N}_{>0}$ and the map $\alpha : M \to k$ defined by the standard embedding on k^* and 0 on $\mathbb{N}_{>0}$. Show that this is a log structure on $\operatorname{Spec} k$, that $(\operatorname{Spec} k, M)$ maps to the standard log point associated to $0^n : \mathbb{N} \to k$, but is incoherent.
  (5) Recall that if $U \subset X$ is open then we can define a log structure by $M =$ local sections of $\mathcal{O}_X$ invertible when restricted to U. Consider X defined by $xy = uv$. Show that the log structure corresponding to the principal open $D(xyuv)$ is log smooth (over a point with trivial log structure), but if you take away any of the divisor it gets worse. Describe these log structures explicitly.
  (6) (some exercises with reducible schemes....)
- Comparing some log structures
  (1) The map $0 \to \mathbb{N}$ induces a map from the log point to the point. Does the analogue hold for the left inverse $\mathbb{N} \to 0$? Compare log differential when possible.
  (2) On $\mathbb{A}^1$ consider the log structures associated to $\mathbb{N} \to k[x]$ defined by $1 \mapsto x$ and $1 \mapsto x^a$ for a a positive integer. Which way do you get a map? compare log differentials.
  (3) Fix a, b nonnegative integers. On $\mathbb{A}^1$ consider the log structures associated to $\mathbb{N}^2 \to k[x]$ defined by $(n, m) \mapsto x^{an+bm}$. Compare the log differentials, especially noting the cases where a or b are 0.
  (4) Investigate when there are maps between the log structures of the previous two exercises and how the log differentials behave.
  (5) Consider the log structure on $\mathbb{A}^2$ associated to $(m, n) \mapsto x^m y^n$. If $P \subset \mathbb{N}^2$ is a submonoid of finite index, how does the associated log structure change? How about log differentials?