LOGARITHMIC STRUCTURE FOR STABLE MAPS RELATIVE TO SIMPLE NORMAL CROSSING DIVISOR

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1. Prerequisites on logarithmic geometry

1.1. **Basic definitions and properties.** Following [Kat89] and [Ogu01], we first recall some basic terminology on logarithmic geometry.

:DefLogStr

1.1.1. *Monoids.* A *monoid* is a commutative semi-group with a unit. We usually use " + " and "0" denote the binary operation and the unit of a monoid. A *morphism between two monoids* is required to preserve the unit.

Let P be a monoid, we can associate a group

$$P^{gp} := \{(a,b)|(a,b) \sim (c,d) \text{ if } \exists s \in P \text{ such that } s+a+d = s+b+c\}.$$

The monoid P is called *integral* if the natural map $P \to P^{gp}$ is injective. And it is called *saturated* if it is integral and satisfies that for any $p \in P^{gp}$, if $n \cdot p \in P$ for some positive integer n then $p \in P$. A monoid P is said to be *fine* if it is integral and finitely generated. A monoid P is called *sharp* if there are no other unit except 0. A nonzero element p in a sharp monoid P is called *irreducible* if $p = a+b$ implies either $a = 0$ or $b = 0$. We denote by $Irr(P)$ the set of irreducible elements in a sharp monoid P. A fine monoid P is called *free* if $P \cong \mathbb{N}^n$ for some positive integer n. A monoid P is called torsion free if the associated group P^{gp} is torsion free. The monoid P is called toric if P is fine, saturated, and torsion free. This is the monoid we will use in this paper.

A morphism $h : Q \to P$ between integral monoids is called *integral* if for any $a_1, a_2 \in Q$, and $b_1, b_2 \in P$ which satisfy $h(a_1)b_1 = h(a_2)b_2$, there exist $a_2, a_4 \in Q$ and $b \in P$ such that $b_1 = h(a_3)b$ and $a_1a_3 = a_2a_4$.

1.1.2. *Logarithmic structures.* Let X be a scheme. A *pre-log structure* on X is a pair $(\mathcal{M}, \exp)$, which consists of a sheaf of monoids $\mathcal{M}$ on the étale site $X_{ét}$ of X, and a morphism of sheaves of monoids $\exp : \mathcal{M} \to \mathcal{O}_X$, called the structure morphism of $\mathcal{M}$. Here we view $\mathcal{O}_X$ as a monoid under multiplication.

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A pre-log structure $\mathcal{M}$ on X is called a *log structure* if $\exp^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$ via exp. We sometimes omite the morphism exp, and only use $\mathcal{M}$ to denote the log structure if no confusion could arise. We call the pair $(X, \mathcal{M})$ a *log scheme*.

Given two log structures $\mathcal{M}$ and $\mathcal{N}$ on X, a *morphism of the log structures* $h : \mathcal{M} \to \mathcal{N}$ is a morphism of sheaves of monoids which compatible with the structure morphisms of $\mathcal{M}$ and $\mathcal{N}$.

Given a pre-log strucutre $\mathcal{M}$ on X, we can associate a log structure $\mathcal{M}^a$ given by

$$\mathcal{M}^a := \mathcal{M} \oplus_{\exp^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*.$$

Consider a morphism of schemes $f : X \to Y$, and a log structure $\mathcal{M}_Y$ on Y. We can define the *pull-back log structure* $f^*(\mathcal{M}_Y)$ to be the log structure associated to the pre-log structure

$$f^{-1}(\mathcal{M}_Y) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X.$$

Consider two log schemes $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y$. A morphism of log schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a pair $(f, f^\flat)$, where $f : X \to Y$ is a morphism of the underlying schemes, and $f^\flat : f^*(\mathcal{M}_Y) \to \mathcal{M}_X$ is a morphism of log structures on X. The morphism $(f, f^\flat)$ is called strict if $f^\flat$ is an isomorphism of log structures. It is called vertical if $\mathcal{M}_X/f^*(\mathcal{M}_Y)$ is a sheaf of groups under the induced monoidal operation.

s:ChartLog

1.1.3. *Charts of log structures.* Let $(X, \mathcal{M})$ be a log scheme, and P a fine monoid. Denote by P_X the constant sheaf of monoid P on X. A chart of $\mathcal{M}$ is a morphism $P_X \to \mathcal{M}$ such that the associated log structure of the composition $P_X \to \mathcal{M} \to \mathcal{O}_X$ is $\mathcal{M}$. The log structure $\mathcal{M}$ is called a fine log structure on X if a chart exists étale locally everywhere on X. If the charts are all given by fine and saturated monoids then $\mathcal{M}$ is called a fs log structure. In this and the following sections, we only consider fine log structures.

Let $\overline{\mathcal{M}} = \mathcal{M}/\mathcal{O}_X^*$ be the quotient sheaf. We call it the characteristic of the log structure $\mathcal{M}$. It is useful to notice that $\overline{f^*(\mathcal{M})} = f^{-1}(\overline{\mathcal{M}})$ for any morphism of schemes $f : Y \to X$. For any closed point $x \in X$, we denote by $\bar{x}$ the separable closure of x. A fine log structure $\mathcal{M}$ is called locally free if for any $x \in X$, we have $\overline{\mathcal{M}}_{\bar{x}} \cong \mathbb{N}^n$ for some positive integer r. Let $\overline{\mathcal{M}}_{\bar{x}}^{gp,tor}$ be the torsion part of $\overline{\mathcal{M}}_{\bar{x}}^{gp}$. The following result is very useful for creating charts.

hartLogStr

Proposition 1.1. [Ols03a, 2.1] *Using the notation as above, there exist an fppf neighborhood $f : X' \to X$ of x, and a chart $\beta : P \to f^*(\mathcal{M})$ such that for some geometric point $\bar{x}' \to X'$ lying over x, the natural map $P \to f^{-1}\overline{\mathcal{M}}_{\bar{x}'}$ is bijective. If $\overline{\mathcal{M}}_{\bar{x}}^{gp,tor} \otimes k(x) = 0$, then such a chart exists in an étale neighborhood of x.*

Remark 1.2. In the following sections, we only work with log structures with toric sharp characteristic over field of characteristic 0. The above proposition implies that in such situation, there is a section of $\mathcal{M}_{\bar{x}} \to \overline{\mathcal{M}}_{\bar{x}}$, which can be lift to a chart étale locally near x.

Consider a morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of fine log schemes. A chart of f is a triple $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ where $P_X \to \mathcal{M}_X$ and $Q_Y \to \mathcal{M}_Y$ are charts of $\mathcal{M}_X$ and $\mathcal{M}_Y$ respectively, and $Q \to P$ is a morphism of monoids such that the following diagram is commutative:

$$\begin{array}{ccc} Q_X & \longrightarrow & P_X \\ \downarrow & & \downarrow \\ f^*(\mathcal{M}_Y) & \longrightarrow & \mathcal{M}_X. \end{array}$$

Similarly, the charts of morphism of fine log schemes exist étale locally by the following result:

Proposition 1.3. [Ols03a, 2.2] *Notations as above, suppose that $Q_Y \to \mathcal{M}_Y$ is a chart. Then étale locally on X, there exist a chart $P_X \to \mathcal{M}_X$ and an injective morphism of monoids $Q \to P$, such that the triple $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ gives a chart for f étale locally on X. If f is a morphism of fs log schemes and if Q is saturated and torsion free, then we can choose P to be also saturated and torsion free in the chart of f.*

Consider a morphism of log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$, with the help of charts, we can describe the log smoothness properties of f that we will use later. The log map f is called log smooth if étale locally, there is a chart $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ of f such that:

(1) $Ker Q^{gp} \to P^{gp}$ and the torsion part of $Coker(Q^{gp} \to P^{gp})$ are finite groups;
(2) the induced map $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}\mathbb{Z}[p]$ is smooth in the usual sense.

The map f is called integral if for every $p \in X$, the induced map $\overline{\mathcal{M}}_{f(\bar{p})} \to \overline{\mathcal{M}}_{\bar{p}}$ is integral. In general, the underlying structure map of a log smooth morphism need not be flat. However, it is shown in [Kat89, 4.5] that the underlying map of a log smooth and integral morphism is flat.

sss:DivLog

1.1.4. *Log structures associated to normal crossing divisors.* This is an important example given in [Kat89, 1.5]. Let X be a smooth scheme, and D is reduced divisor in X with normal crossings. We define a fine log structure on X:

$$\mathcal{M}^D := \{g \in \mathcal{O}_X | g \text{ is invertible outside } D\} \subset \mathcal{O}_X.$$

For each point $p \in D$, let $\{g_i\}_{i=1}^n$ be the set local coordinates near $\bar{p}$, such that D is given by the vanishing of $g_1 \cdots g_n$. Then the log structure $\mathcal{M}^D_{\bar{p}}$ is generated by $\{\log g_i\}_{i=1}^n$, where $\log g_i$ denote the pre-image of g_i in the log structure. Thus étale locally near $\bar{p}$ we have a chart

$$\mathbb{N}^n \to \mathcal{M}^D \quad e_i \mapsto g_i,$$

where e_i is the standart generators of $\mathbb{N}^n$. And the above chart gives an isomorphism $\mathbb{N}^n \cong \overline{\mathcal{M}}^D_{\bar{p}}$. We see that the log structure $\mathcal{M}^D$ is locally free, and its rank at a point p equals the number of components of D at p.

Consider the case when D is a finite union of reduced divisors in X with normal crossings. Denote by $D = \coprod_j D_j$. The same definition as above gives a log structure $\mathcal{M}^D$ on X. On the other hand, for each D_j, we associate a log structure $\mathcal{M}^{D_j}$ on X as above. We have decomposition:

$$\mathcal{M}^D \cong \sum_j \mathcal{M}^{D_j},$$

where the sum is taking over $\mathcal{O}_X^*$.

We can assume that X is an algebraic stack, and D is a finite union of reduced divisors in X with normal crossings. Then we can still define the locally free log structure $\mathcal{M}^D$ on X using smooth topology, and the decomposition of $\mathcal{M}^D$ into amalgamated sum of $\mathcal{M}^{D_j}$ still holds.

sss:LogPt

1.1.5. *Logarithmic point.* Let $S = \operatorname{Spec} k$, and $\mathcal{M}$ a rank n locally free log structure on S. The pair $(S, \mathcal{M})$ is called a log point if $\mathcal{M} \cong \mathbb{N}^n \oplus k^*$, and the structure morphism $\mathcal{M} \to k$ is given by $e_i \mapsto 0$, where the element e_i is the standard generator of $\mathbb{N}^n$.

In general, a locally free log structure does not imply that $(S, \mathcal{M})$ is a log point. However, when k is a separably closed field, then any pair $(S, \mathcal{M})$ with $\mathcal{M}$ locally free is a log point.

s:LogStack

1.2. **Olsson's Log Stacks.** We follow [Ols03a] to introduce the algebraic stack parametrizing log schemes. Let us fix a base scheme S, and consider an algebraic stack $\mathcal{X}$ in the sense of [Art74], which means that the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable and of finite type, and there exists a surjective smooth morphism $X \to \mathcal{X}$ from a scheme. Now we can define a fine log structure $\mathcal{M}_{\mathcal{X}}$ on $\mathcal{X}$ by repeating the definitions in 1.1.2 and 1.1.3 but using lisse-étale site instead of the étale site. See [Ols03a, Section 5] for details.

For any S-scheme T, and an arrow $g : T \to \mathcal{X}$, we obtain a fine log structure $g^*(\mathcal{M}_{\mathcal{X}})$ on the lisse-étale site $T_{lis\text{-}et}$ of T. It is shown in [Ols03a, 5.3] that such $g^*(\mathcal{M}_{\mathcal{X}})$ is isomorphic to a unique fine log structure on the étale site T_{et} of T. By abusing of notations, we still use $g^*(\mathcal{M}_{\mathcal{X}})$ denote this new log structure on T. By pull-back the log structure $\mathcal{M}_{\mathcal{X}}$, we define a functor from $\mathcal{X}$ to the category of fine log schemes over S. The stack $\mathcal{X}$ associated with this functor is called a log stacks in [Kat00]. A fine log scheme $(X, \mathcal{M}_X)$ can be naturally viewed as a log algebraic stack.

Still consider the log algebraic stack $(\mathcal{X}, \mathcal{M}_X)$. We define a fibered category $\mathcal{L}og_{(\mathcal{X},\mathcal{M}_{\mathcal{X}})}$ over $\mathcal{X}$. Its objects are pairs $(g : X \to \mathcal{X}, g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X)$, where g is a map from scheme X to $\mathcal{X}$, and $g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X$ is a morphism of fine log structures on X. An arrow $(g : X \to \mathcal{X}, g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X) \to (h : Y \to \mathcal{X}, h^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_Y)$ is a strict morphism of log schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$, such that the underlying map $X \to Y$ is a morphism over $\mathcal{X}$, and we have the following commutative diagram:

$$\begin{array}{ccc} (X, \mathcal{M}_X) & \longrightarrow & (Y, \mathcal{M}_Y) \\ \downarrow & & \downarrow \\ (X, g^*(\mathcal{M}_{\mathcal{X}})) & \longrightarrow & (Y, h^*(\mathcal{M}_{\mathcal{X}})). \end{array}$$

Remark 1.4. In fact, an object $(g : X \to \mathcal{X}, g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X)$ can be viewed as a morphism of log stacks $(X, \mathcal{M}_X) \to (\mathcal{X}, \mathcal{M}_{\mathcal{X}})$. Roughly speaking, the stack $\mathcal{L}og_{(\mathcal{X},\mathcal{M}_{\mathcal{X}})}$ parametrizes log schemes over $(\mathcal{X}, \mathcal{M}_{\mathcal{X}})$. For the definition of morphisms of log stacks, we refer to [Ols03a], and this one is compatible with the definition of morphisms between log schemes.

Proposition 1.5. [Ols03a, 5.9] *The fibered category $\mathcal{L}og_{(\mathcal{X},\mathcal{M}_{\mathcal{X}})}$ is an algebraic stack locally of finite presentation over $\mathcal{X}$.*

2. Logarithmic curves and their stacks

In this section, we define log pre-stable curves in our sense, and show that the stack $\mathfrak{M}_{g,n}^{pre}$ parametrizing log pre-stable curves of genus g and n marked points in our sense is an open substack of a log stack in the sense of [Ols03a], hence is algebraic in the sense of [Art74, 5.1].

2.1. **The canonical log structure on pre-stable curves.** We first introduce the canonical log structure on pre-stable curves. For details, we refer the reader to [Kat00], [S.M95], and [Ols07].

Let $\mathfrak{M}_{g,n}$ be the stack parametrizing genus g pre-stable curves with n marked points, and let $\mathfrak{C}_{g,n}$ be the universal family over $\mathfrak{M}_{g,n}$. Denote by $\{\Sigma_i : \mathfrak{M}_{g,n} \to \mathfrak{C}_{g,n}\}_{i=1}^{n})$ the n sections. The locus $\mathfrak{M}_{g,n}^{sing}$ in $\mathfrak{M}_{g,n}$ which parametrizes singular curves is a divisor with normal crossings on $\mathfrak{M}_{g,n}$. Hence by [Kat89, 1.5], there is a canonical log structure $\mathcal{M}_{\mathfrak{M}_{g,n}}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}$ on $\mathfrak{M}_{g,n}$, which is defined on the smooth topology in the sense of [Ols03a]. Note that the n sections and the pre-image of $\mathfrak{M}_{g,n}^{sing}$ in $\mathfrak{C}$ also give a divisor with normal crossings on $\mathfrak{C}_{g,n}$. Hence we obtain a log structure $\mathcal{M}_{\mathfrak{C}}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}$ on $\mathfrak{C}$. There is a natural log smooth map $(\mathfrak{M}_{g,n}, \mathcal{M}_{\mathfrak{M}_{g,n}}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}) \to (\mathfrak{C}, \mathcal{M}_{\mathfrak{C}}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}})$ whose underlying map is given by the family $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$.

Given any family $C \to S$ of usual pre-stable curves of genus g, with n marked points, we have the following cartesian diagram:

$$\begin{array}{ccc} C & \longrightarrow & \mathfrak{C}_{g,n} \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ S & \longrightarrow & \mathfrak{M}_{g,n}. \end{array}$$

Pulling back the canonical log structures on $\mathfrak{C}_{g,n}$ and $\mathfrak{M}_{g,n}$, we obtain canonical log structures $\mathcal{M}_C^{C/S}$ and $\mathcal{M}_S^{C/S}$ on C and S respectively, and a natural log smooth map $\pi : (C, \mathcal{M}_C^{C/S}) \to (S, \mathcal{M}_S^{C/S})$.

Using the notation as above, the log structure $\mathcal{M}_{\mathfrak{M}_{g,n}}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}$ is locally free, hence the canonical log structure $\mathcal{M}_S^{C/S}$ is also locally free. Then for any closed point $s \in S$, we have $\overline{\mathcal{M}}_{S,\bar{s}}^{C/S} \cong \mathbb{N}^m$, and this m equal to the number of the nodes in the fiber $C_{\bar{s}}$. In fact we have a one-to-one correspondence between the m factors of the monoid $\mathbb{N}^m$ and the nodes on the fiber.

s:LocalCan

2.2. **Local description of the canonical log structure on pre-stable curves.** By [Ols03a, 2.1], we can shrink S if necessary, and assume that we have a global chart $\mathbb{N}^m \to \mathcal{M}_S^{C/S}$ given by $\overline{\mathcal{M}}_{S,\bar{s}}^{C/S}$. We denote $\{e_i\}_{i=1}^m$ be the standard generators of $\mathbb{N}^m$.

Consider a closed point $p \in C_{\bar{s}}$ in the fiber. If p is a smooth non-marked point, then we have an étale neighborhood $\bar{p} \in U \subset C$, such that $\mathcal{M}_C^{C/S}|_U = \pi^*(\mathcal{M}_S^{C/S})|_U$.

When p is a marked point given by the section Σ_i, then consider an étale neighborhood $p \in U$ which contains only smooth points of C over S, and no other markings. We have the log structure

$$\mathcal{M}_C^{C/S}|_U = \pi^*(\mathcal{M}_S^{C/S})|_U \oplus_{\mathcal{O}_U^*} \mathcal{M}^{\Sigma_i}|_U,$$

where the log structure $\mathcal{M}^{\Sigma_i}$ is given by the section Σ_i, which locally has a chart $\mathbb{N} \to \mathcal{M}^{\Sigma_i}$. Hence we have a chart $\mathbb{N}^m \oplus \mathbb{N} \to \mathcal{M}_C^{C/S}|_U$.

Finally, let us assume p is a node. Then there is an étale neighborhood U of $\bar{p}$, which contains no other nodes and marked points. We have a special element $e_j \in \{e_i\}_{i=1}^m$, with the following chart:

$$\begin{array}{ccc} \mathbb{N}^{m-1} \oplus \mathbb{N}^2 & \longrightarrow & \mathcal{M}_C^{C/S}|_U \\ {\scriptstyle(id,\Delta)}\uparrow & & \uparrow{\scriptstyle\pi^\flat} \\ \mathbb{N}^{m-1} \oplus \mathbb{N} & \longrightarrow & \pi^*(\mathcal{M}_S^{C/S})|_U. \end{array}$$

Here on the bottom, the monoids $\mathbb{N}^{m-1}$ and $\mathbb{N}$ are generated by $\{e_i\}_{i\neq j}$ and e_j respectively, and on the top we assume that a and b are the standard generators of the monoid $\mathbb{N}^2$. The map (id, Δ) is given by the identity on $\mathbb{N}^{n-1}$ and the diagnonal map $\Delta : e_j \mapsto a + b$.

Definition 2.1. We identify e_j with its image in the log structre, and call it an element in $\mathcal{M}_S^{C/S}$ smoothing the node p, or simply an element smoothing p.

Note that two elements smoothing a same node are differ by an invertible function near the node, therefore they induce the same element in the characteristic monoid $\overline{\mathcal{M}}_S^{C/S}$.

For each node p_i over s, we fix an element e_i smoothing it. Denote by $\bar{e}_i$ the image of e_i in $\overline{\mathcal{M}}_S^{C/S}$. Let $Irr(\overline{\mathcal{M}}_{S,\bar{s}})$ be the set of irreducible elements in the monoid $\overline{\mathcal{M}}_{S,\bar{s}}$. In fact we have $\{\bar{e}_i\}_{i=1}^m = Irr(\overline{\mathcal{M}}_{S,\bar{s}})$, and a natural map:

$$s_{C_{\overline{s}}} : \{\text{nodes in } C_{\overline{s}}\} \longrightarrow Irr(\overline{\mathcal{M}}_{S,\bar{s}})$$

given by $p_i \mapsto$ (the element e_i smoothes p_i). It was shown in [Kat00] that this map is a one-to-one correspondance. This means that all nodes in the fiber are smoothed independently.

em:Special *Remark* 2.2. The bijection $s_{C_{\overline{s}}}$ implies that the canonical log structures $(\mathcal{M}_S^{C/S}, \mathcal{M}_C^{C/S})$ is special in the sense of [Ols03b, 2.6].

ode-To-Log *Remark* 2.3. The one to one correspondance $s_{C_{\overline{s}}}$ associates to each node p_i a unique sub-log structure $\mathcal{N}_i \subset \mathcal{M}_S$ generated by e_i. In an étale neighborhood of $\bar{s}$, it was shown in [Kat00] that

$$\mathcal{M}_S \cong \mathcal{N}_1 \oplus_{\mathcal{O}_S^*} \cdots \oplus_{\mathcal{O}_S^*} \mathcal{N}_m.$$

CanLog

2.3. **The canonical log structure at node.** We give a local description of the relation between canonical log structure and the underlying structure at the nodes as in [Kat00, Section 3]. Let A be a local neotherian henselian ring, and s an element in the maximal ideal m_A of A. Let R be the henselization of $A[x, y]/(xy - s)$ at the ideal generated by x, y and m_A. We still use x, y to denote the corresponding elements in R.

erDesCurve **Lemma 2.4.** [Kat00] *With the notation as above, we have the following: Given $x', y' \in R$ such that $x'y' \in A$ and $(x', y', m_A) = (x, y, m_A)$ (equality of ideals in R). Then there exist units $u_x, u_y \in R^*$ with $u_x u_u \in A$ such that $x' = u_x x$ and $y' = u_y y$ (or $y' = u_x x$ and $x' = u_y y$).*

Consider the local family $\mathrm{Spec}R \to \mathrm{Spec}A$, the canonical log structure $(\mathcal{M}_R, \mathcal{M}_A)$ is given by the following commutative diagram of prelog structures.

$$\begin{array}{ccc} \mathbb{N}^2 & \xrightarrow{(e_1,e_2)\mapsto(x,y)} & R \\ {\scriptstyle\Delta}\uparrow & & \uparrow \\ \mathbb{N} & \xrightarrow{e\mapsto s} & A \end{array}$$

where e_1, e_2 (resp. e) are the standard generators of $\mathbb{N}^2$ (resp. $\mathbb{N}$), and $\Delta : e \mapsto e_1 + e_2$ is the diagonal map. For convenience, we sometimes use $\log x, \log y$ and $\log s$ denote the image of e_1, e_2 and e in the corresponding log structures.

Corollary 2.5. [Kim, 3.6.2] *We use the notation as above, and let l be a positive integer. Then there is a unique pair γ_x, γ_y in $\mathcal{M}_R$, which will be denoted by $l \log x, l \log y$ respectively, such that $\gamma_x + \gamma_y \in \mathcal{M}_A$ and $\exp(\gamma_x) = x^l, \exp(\gamma_y) = y^l$*

ss:UnivCan 2.4. **Universal property of canonical log structure.** Next we introduce another description of the canonical log structre. In fact, this is the description given in [Kat00] and [Ols07, 3.9,3.10], except that in our case, we do not introduce orbifold structure.

Now we consider a new log structure on the fiber $\mathcal{M}_C^{\sharp C/S}$ which is obtained by removing the log structure corresponding to the markings. This is equivalent to require that the log structure near the marked points is pull back of the log structures from the base. By our description of canonical log structures, we have the relation

$$\mathcal{M}_C^{C/S} = \mathcal{M}_C^{\sharp C/S} \oplus_{\mathcal{O}_C^*} (\sum_j \mathcal{M}^{\Sigma_j}).$$

And we still have a log map $\pi^\sharp : \mathcal{M}_C^{\sharp C/S} \to \mathcal{M}_S^{C/S}$. This map is log smooth, proper, integral, vertical, and special (see remark 2.2). In fact, we have the following universal property.

UnivCanLog **Lemma 2.6.** *For any pair of fine log structures $(\mathcal{M}'_C, \mathcal{M}_S)$ over the family of prestable curves $C \to S$, such that the log map $(C, \mathcal{M}'_C) \to (S, \mathcal{M}_S)$ is log smooth, proper, integral and vertical, we have a unique pair of maps $\mathcal{M}_C^{\sharp C/S} \to \mathcal{M}'_C$ and $\mathcal{M}_S^{C/S} \to \mathcal{M}_S$ fitting in the following cartesian diagram of fine log schemes:*

$$\begin{array}{ccc} (C, \mathcal{M}'_C) & \longrightarrow & (C, \mathcal{M}_C^{\sharp C/S}) \\ \downarrow & & \downarrow \\ (S, \mathcal{M}_S) & \longrightarrow & (S, \mathcal{M}_S^{C/S}), \end{array}$$

Proof. See [Ols07], and [Ols03b, 2.7] for a proof.

Remark 2.7. We remark that the canonical log structure $\mathcal{M}_S^{\sharp C/S}$ does not depend on the markings.

2.5. **Log curves.** With the description above, we are able to introduce the log structure on curves that we are interested in.

DefLogC1 **Definition 2.8.** A map of fine log schemes $(C, \mathcal{M}_C) \to (S, \mathcal{M}_S)$ with sections $\{\Sigma_i\}_{i=1}^n$ is called a genus g log curve with n-markings if

(1) the family $C \to S$ with $\{\Sigma_i\}$ is the usual prestable curve of genus g and n-markings;
(2) the log structure $\mathcal{M}_C$ is of the form $\mathcal{M}_C = \mathcal{M}'_C \oplus_{\mathcal{O}_C^*} (\sum_j \mathcal{M}^{\Sigma_j})$;
(3) the log map $(C, \mathcal{M}_C) \to (S, \mathcal{M}_S)$ comes from a log smooth, integral vertical map $(C, \mathcal{M}'_C) \to (S, \mathcal{M}_S)$ plus the log structure $\mathcal{M}^{\Sigma_i}$ given by the markings.

By lemma 2.6, we have an equivalent definition of log curves using the canonical log structure.

DefLogC2 **Definition 2.9.** A genus g, log curve with n-marked points over a scheme S is given by the following data $(C \to S, \{\Sigma\}_{i=1}^n, \mathcal{M}_S^{C/S} \to \mathcal{M}_S)$, where

(1) $(C \to S, \{\Sigma\}_{i=1}^n)$ is a usual family of pre-stable curves of genus g, n-markings;
(2) $\mathcal{M}_S^{C/S} \to \mathcal{M}_S$ is a morphism of fine log structures.

When no confusion would arise, we denote by $(C \to S, \mathcal{M}_S)$ the log curves in the definition for short. We use $\mathcal{M}_C$ for the log structure on the curves in the above definition 2.8.

FrLogCurve

2.6. **Log pre-stable curves.**

ogPreCurve **Definition 2.10.** A log curve $(C \to S, \mathcal{M}_S)$ is called log pre-stable if for any geometric point $\bar{s} \in S$, the characteristic $\overline{\mathcal{M}}_{S,\bar{s}}$ is a toric sharp monoid.

For simplicity, we consider the case where S is a geometric point. Note that we have a map on the level of characteristic $\overline{\mathcal{M}}_S^{C/S} \to \overline{\mathcal{M}}_S$. Since the log structure $\overline{\mathcal{M}}_S^{C/S}$ is locally free, we fix $\overline{\mathcal{M}}_S^{C/S} \cong \mathbb{N}^m$, and denote by $\{e_i\}_{i=1}^m$ the set of all irreducible elements in $\overline{\mathcal{M}}_S^{C/S}$.Consider the map on the level of characteristic $\bar{\psi} : \overline{\mathcal{M}}_S^{C/S} \to \overline{\mathcal{M}}_S$. By remark 2.3, let p be the node corresponds to e_i. We call $\bar{\psi}(e_i)$ *the element smoothes p in $\overline{\mathcal{M}}_S$*. Later for convenience, we will identify e_i with its image $\bar{\psi}(e_i)$ in $\overline{\mathcal{M}}_S$.

CurveOpen *Remark* 2.11. We note that the log pre-stable property in 2.10 is an open condition. Given a log curves $(C \to S, \mathcal{M}_S)$, consider a closed point $s \in S$. Assume that the characteristic $\overline{\mathcal{M}}_{S,\bar{s}}$ is toric sharp, then by 1.1, in an étale neighborhood of s there is a section $\overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$, which gives a chart for $\mathcal{M}_S$. By [Ols03a, 3.5], the condition that the characteristic is toric and sharp is constructible and stable under generalization. Hence is an open condition.

2.7. **The stack of log curves.** We recall the definition of arrows log curves as in [Che].

soLogCurve **Definition 2.12.** Given two log curves $(C \to S, \mathcal{M}_S)$ and $(C' \to S, \mathcal{M}'_S)$ over S. Denote $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ the log structure on C and C' associated to the two log curves respectively. An isomorphism between the above two log curves is a pair (ρ, θ) such that

(1) $\theta : (S, \mathcal{M}_S) \to (S, \mathcal{M}'_S)$ and $\rho : (C, \mathcal{M}_C) \to (C', \mathcal{M}_{C'})$ are isomorphisms of log schemes;
(2) the underlying map $\underline{\theta} : S \to S$ is the identity, and $\underline{\rho} : C \to C'$ is an isomorphism of usual prestable curves over S;
(3) the pair (ρ, θ) fit in the following commutative diagram:

$$\begin{array}{ccc} (C, \mathcal{M}_C) & \xrightarrow{\rho} & (C', \mathcal{M}_{C'}) \\ \downarrow & & \downarrow \\ (S, \mathcal{M}_S) & \xrightarrow{\theta} & (S, \mathcal{M}'_S). \end{array}$$

Denote by $\mathfrak{M}_{g,n}^{log}$ the fibered category over $\mathbb{C}$ parametrizing log curves with the arrow defined above. As in [Che], the fibered category $\mathfrak{M}_{g,n}^{log}$ forms an algebraic stack in the sense of [Art74]. Denote by $\mathfrak{M}_{g,n}^{pre}$ the substack of $\mathfrak{M}_{g,n}^{log}$ parametrizing log prestable curves. Then by remark 2.11, we have the following:

CurveStack **Corollary 2.13.** *The fibered category $\mathfrak{M}_{g,n}^{pre}$ is an open substack in $\mathfrak{M}_{g,n}^{log}$, hence is algebraic.*

3. Log maps to smooth pairs

Definition 3.1. We call (X, D) a *smooth pair*, if X is a smooth projective variety, and $D = \cup_{i=1}^k D_i$ is a union of smooth divisors with normal crossing singularities. Hence we have a log structure on X associated to the smooth divisor, denoted by $\mathcal{M}_X$. In the following, when we mention a log map to the smooth pair, we mean the log maps to the log scheme $(X, \mathcal{M}_X)$.

Remark 3.2. For simplicity, in the rest of the paper, we fix the smooth pair (X, D), and assume that that the intersection of any two divisors D_i and D_j are connected. But it will be not hard to see that all our construction and results work for D with arbitrary simple normal crossings.

Note that locally around a point p in D we have a canonical isomorphism $\mathbb{N}^m \to \overline{\mathcal{M}}_{X,p}$. Denote by δ_i the standard generator of $\overline{\mathcal{M}}_{X,p}$ which corresponds to the smooth divisor D_i.

efn:LogMap **Definition 3.3.** A log map over S is given by the data $(C \to S, \mathcal{M}_S, f)$, where $(C \to S, \mathcal{M}_S)$ is a log curve, and $f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$ is a log map.

LogMapChar

3.1. **Log morphism on the level of characteristic.** Consider a log map $\xi = (\pi : C \to S, \mathcal{M}_S, f)$ as in definition 3.3, where $S = \mathrm{Spec} k$ is a geometric point and $(C \to S, \mathcal{M}_S)$ is a log prestable curve. Consider a point $p \in C$, which sits in an irreducible component Z. Then on the level of characteristic, we have a map

$$\bar{f}^\flat_p : f^*(\overline{\mathcal{M}}_X)_p \to \overline{\mathcal{M}}_{C,p}.$$

Assume that the image $f(p)$ lies in the intersection of divisors $\{D_i\}_{I_p}$, where $I_p \subset \{1, \cdots, k\}$. Denote by $m_p = |I_p|$. Then the free monoid $\pi^*(\overline{\mathcal{M}}_X)_p$ is generated by the pull-back of δ_i for $i \in I_p$. Abusing of notations, we assume that $\{\delta_i\}$ is a set of generators of $\pi^*(\overline{\mathcal{M}}_X)_p$. As in [Che], we have three different cases.

First consider the case p is a smooth non-marked point. By the description in definition 2.8, we have $\bar{f}^\flat(\delta_i) = e_i \in \overline{\mathcal{M}}_S$, for $i \in I_p$. We call it the i-th degeneracy at p. By proposition 1.1, the smooth point in Z will all have the same i-th degeneracy. Therefore, we call the element e_i the i-th degeneracy of Z.

If $i \notin I_p$ for some $p \in Z$, we define the i-th degeneracy of Z to be $e_i = 0$ in $\overline{\mathcal{M}}_S$. Note that in this case, the component Z does not map to the divisor D_i.

Definition 3.4. The k-tuple $(e_i)_{i=1}^k$ is called the degeneracy of Z where e_i is the i-th degeneracy of Z. Denote by $I_Z = \{\ i \mid e_i \neq 0\}$.

Next, we consider the case where p is a marked point. Since locally at p, we have $\mathcal{M}_C \cong \pi^*\mathcal{M}_S \oplus_{\mathcal{O}^*_C} \mathcal{N}$, where $\mathcal{N}$ the the canonical log structure associated to the marked point p. Then on the level of characteristic, we have

u:MarkedPt
$$\bar{f}^\flat(\delta_i) = e_i + c_{i,p} \cdot \sigma_p, \tag{3.1.1}$$

where $e_i \in \overline{\mathcal{M}}_S$, and σ_p is the generator of $\overline{\mathcal{N}}$, and $c_{i,p}$ is a positive integer.

egMarkedPt *Remark* 3.5. When we generalize the equation (3.1.1) to the nearby smooth points, the element σ_p will be an ivertible element from the structure sheaf. Thus, the element e_i is the i-th degeneracy of the component Z.

Definition 3.6. We call $c_{i,p}$ the i-th contact order of f at p, and the sum $c_p = \sum_{i=1}^k c_{i,p}$ the contact order at p.

Lemma 3.7. [1] *Openness for contact order*

Finally, let us consider the case where p is a node connecting Z with another irreducible component Z'. Let e be the element in $\overline{\mathcal{M}}_S$ smoothing the node p, and $\log x_p, \log y_p$ are

[1]Go Back to this later

the element in $\overline{\mathcal{M}}_C$ correspond to the local coordinates of the two components Z and Z' respectively as in subsection 2.3. Then we have the equation in $\overline{\mathcal{M}}_C$:

:NodeCurve (3.1.2)
$$e = \log x_p + \log y_p.$$

Thus, without loss of generality we can assume that

equ:Node (3.1.3)
$$\bar{f}^\flat(\delta_i) = e_i + c_{i,p} \cdot \log x_p,$$

where $c_{i,p}$ is a positive integer.

n:DistNode **Definition 3.8.** The integer $c_{i,p}$ is called the i-th contact order of f at the node p. If $c_{i,p} \neq 0$, then p is called an i-distinguished node.

DegNodeLow *Remark* 3.9. The same argument as in remark 3.5 shows that e_i is the degeneracy of Z. We call that Z is the i-lower component of p, and Z' is the i-upper component of p.

em:DegJump **Lemma 3.10.** *Using notations as above, the i-th degeneracy of Z' is $e_i + c_{i,p} \cdot e$.*

ss:AdGraph 3.2. **Admissible graph.**

eightGraph **Definition 3.11.** A k-weighted graph G is a connected graph with the following data:

(1) a set of vertices $V(G)$, such that for each $v \in V(G)$ we associate a k-tuple $(e_{v,i})_{i=1}^k$ called the weights of v, where $e_{v,i}$ is either 0 or a variable;
(2) A set of edges $E(G)$, such that for each $l \in E(G)$ we associate a k-tuple of non-negative integers $(c_{l,i})_{i=1}^k$ called the contact orders of l, and a variable e_l called the weight of l.

These data satisfies the only condition that if the edge l is a loop, then $c_{l,i} = 0$ for all i. If the contact orders of an edge l are all zero, then l is called the non-distinguished edge. Two vertices is called adjacent if they are connected by an edge.

rientation **Definition 3.12.** Consider a k-weighted graph G as in the above definition. An orientation on G is a set of partial orders $\{\leq_i\}_{i=1}^k$ on the set of vertices $V(G)$, such that for any two vertices v_1 and v_2 connecting by an edge $l \in E(G)$ we have

(1) If $c_{l,i} \neq 0$, then we have either $v_1 \leq_i v_2$ or $v_2 \leq_i v_1$;
(2) If $c_{l,i} = 0$, then we have both $v_1 \leq_i v_2$ and $v_2 \leq_i v_1$;
(3) If for a vertex v we have $e_{v,i} = 0$, then for any other adjacent vertex v' of v we have $v \leq_i v'$.

Consider an edge $l \in E(G)$, and its two end vertices v_1 and v_2. If $v_1 \leq_i v_2$, we call v_1 the i-initial vertex of l, and v_2 the i-end vertex of l. An i-path is a squence of edges $(l_1, l_2, \cdots, l_m)$ such that the i-end vertex of l_j is the i-initial vertex of l_{j+1}. Such i-path is called an i-loop if the i-initial vertex of l_1 is the i-end vertex of l_m. A vertex $v \in V(G)$ is called i-minimal (respectively i-maximal) if it is not the i-end (respectively i-initial) vertex of any edge. Thus, any vertex v with the zero i-th weight is i-minimal.

Consider a k-weighted oriented graph G as in the above definition. For each edge $l \in E(G)$ and its i-initial vertex v_1 and i-end vertex v_2, we can associate an equation

u:GraphEqu (3.2.1)
$$h_{l,i}: \quad e_{v_2,i} = e_{v_1,i} + c_{l,i} \cdot e_l$$

Consider the monoid

$$M = \Big\langle e_{v,i}, e_l \,|\, \text{for all } i \in \{1, \cdots, k\}, \text{ and } l \in E(G), \text{ with all the relations } h_{l,i} \Big\rangle.$$

Denote by $M(G)$ the torification of M.

fn:AdGraph **Definition 3.13.** We call $M(G)$ the associated monoid of the k-weighted oriented graph G.

For each k-weighted oriented graph G, we would like to focus on the non-zero weights. We associated a new graph G^{deg} as follows:

(1) Identify all vertices in G with zero weights.
(2) Identify all vertices and their weights that are connected by a path formed by non-distinguished edges.
(3) Contract all non-distinguished edges.

By (2) above, we can define the weights of vertices in G^{deg} given by the weights from the corresponding vertices in G. The weights and contact orders of edges in G^{deg} can be obtained from the corresponding edges in G, since we only contract non-distinguished edges. Note the G^{deg} is a k-weighted graph.

Definition 3.14. The graph G^{deg} is called the contracted graph of G.

Proposition 3.15. *Using the notations as above, we have*

(1) *The orientation $\{\leq_i\}$ in G induces an orientation in G^{deg}. Thus G^{deg} is a k-weighted oriented graph.*
(2) *We have a canonical isomorphism $M(G) \cong M(G^{deg}) \oplus \mathbb{N}^m$, where m is the number of non-distinguished edges in G.*

Proof. The first statement follows from the definition 3.12 and the construction of G^{deg}. To prove the second statment, we first notice that in the construction of G^{deg}, we identify the weights of any two vertices connected by a non-distinguished path, which is equivalent to the equation 3.2.1 given by the non-distinguished edges from the path. It is clear that we have an injection $M(G^{deg}) \to M(G)$. Denote by $\{e_i\}_{i=1}^m$ the set of weights of the non-distinguished edges. First notice that none of the elements in $\{e_i\}$ is involved in the equations 3.2.1. Thus, these elements give the part $\mathbb{N}^m$. Note that $M(G)$ is generated by $M(G^{deg})$ and $\{e_i\}$. It follows that $M(G) \cong M(G^{deg}) \oplus \mathbb{N}^m$. □

Corollary 3.16. *The graph G is admissible if and only if G^{deg} is admissible.*

Proof. This follows directely from the above lemma.

Note that we can identify the weights $e_{v,i}$ and e_l with the element in $M(G)$. Denote by $N(G)$ the submonoid of $M(G)$ generated by the weights $e_{v,i}$ and e_l.

htInMonoid **Lemma 3.17.** *Using the notations as above, we have*

(1) *The associated group $M(G)^{gp}$ is generated by all the weights from vertices and edges.*
(2) *The saturation of $N(G)$ is $M(G)$, namely for any $a \in M(G)$, there exists $b \in N(G)$ an a positive integer m such that $b = m \cdot a$.*

Proof. This follows from the definition of $M(G)$. □

Definition 3.18. The Graph is admissible if $M(G)$ is a sharp monoid.

Corollary 3.19. *If G is admissible, then there is no i-loop in G for any i*

Proof. If there is an i-loop, then $M(G)$ fails to be sharp, which contradicts the assumption. □

3.3. **Graph associated to log maps.** Consider a log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S. We construct a dual graph G_ξ of ξ as follows:

(1) The vertices of G_ξ is given by the set
$$V(G_\xi) = \big\{\ v \mid v \text{ is an irreducible component of } C\big\}.$$
For each $v \in V(G_\xi)$, we associate a k-tuple of weights $(e_{v,i})_{i=1}^k$ such that $e_{v,i}$ is a variable if v degenerate into D_i, and 0 otherwise.

(2) The edges of G_{xi} is given by the set
$$E(G_\xi) = \big\{\ l \mid l \text{ is a node of } C\big\}.$$
For each $l \in E(G_\xi)$, we associate a k-tuple of non-negative integers $(c_{l,i})_{i=1}^k$ and a variable e_l, such that $c_{l,i}$ is the i-th contact order of the node l as in definition 3.8.

(3) For each $i \in \{1, 2, \cdots, k\}$, we associate an partial orders as follows. Let $l \in E(G_\xi)$ be a node connecting two irreducible components $v_1, v_2 \in V(G_\xi)$. Then $v_1 \leq_i v_2$ is v_1 is i-lower component and v_2 is the i-upper component of l as in remark 3.9.

Lemma 3.20. *The graph G_ξ is a k-weighted, oriented graph.*

Proof. This can be proved by checking the definition directly using result from subsection 3.1. □.

Denote by G_ξ^{deg} the contracted graph associated to G_ξ

Definition 3.21. We call G_ξ the dual graph, and G_ξ^{deg} the degeneracy graph of ξ.

Consider a node $l \in E(G_\xi)$, denote by e_l' the element in $\overline{\mathcal{M}}_S$ which smoothes l, and e_l the weight of l. Then consider a irreducible component $v \in V(G_\xi)$, denote by $e_{v,i}'$ the i-th degeneracy of v in ξ, and $e_{v,i}$ the i-th weight of v. We define a correspondance

CanMapChar (3.3.1)
$$e_l \mapsto e_l' \quad \text{and} \quad e_{v,i} \mapsto e_{v,i}'$$

By lemma 3.10 and 3.17, the above correspondance induces a canonical morphism of groups
$$\phi : M(G_\xi)^{gp} \to \overline{\mathcal{M}}_S^{gp}$$

op:CanChar **Proposition 3.22.** *Assume that $\mathcal{M}_S$ is toric, then the map ϕ induces a canonical morphism of monoids $M(G_\xi) \to \overline{\mathcal{M}}_S$.*

Proof. It is enough to show that the image of $M(G_\xi)$ under ϕ is in $\overline{\mathcal{M}}_S$. First notice that the image of $N(G_\xi)$ under ϕ is in $\overline{\mathcal{M}}_S$. Since $\overline{\mathcal{M}}_S$ is saturated, the statement follows from lemma 3.17. □

We still use ϕ denote the induced map $M(G_\xi) \to \overline{\mathcal{M}}_S$.

:CanLogMap **Definition 3.23.** The log map ξ is called canonical if the induced canonical map ϕ is an isomorphism.

3.4. **Log stable maps.**

:DataGamma **Conventions.** *In the rest of the paper, we fix the discrete data $\Gamma = (\beta, g, N, M, \boldsymbol{c})$ where*

(1) *$\beta \in H^2(X, \mathbb{Z})$ is a curve class in X;*
(2) *g is a non-negative integer;*
(3) *N is a finite ordered set which we may take to be $\{1, \cdots, n\}$;*

(4) *for each* $p \in N$ *we associate a* k*-tuple of non-negative integers* $(c_{p,i})_{i=1}^k$ *such that*

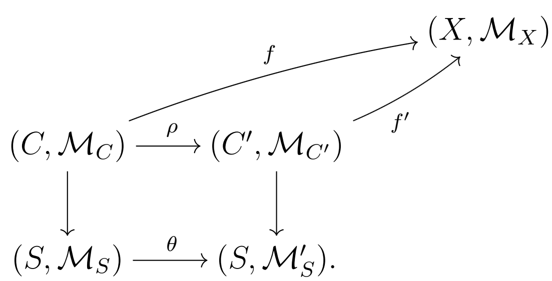
$$\sum_{p\in N} c_{p,i} = \beta.D_i \quad \textit{for any } i.$$

gStableMap **Definition 3.24.** The log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S is called Γ-log stable if

1. The source curve $(C \to S, \mathcal{M}_S)$ is a log pre-stable curve of genus g with marked points numbered by N.
2. The underlying map f is stable in the usual sense.
3. the curve class $f_*(C) = \beta$.
4. The contact orders of each marked point $p \in N$ is given by $(c_{p,i})_{i=1}^k$.
5. The log map ξ is canonical.

A log map ξ' over a scheme T is called Γ-log stable if its geometric fibers are all Γ-log stable. Since we fix all the discrete data, we will omit Γ in the rest of the paper..

Definition 3.25. Consider two log maps $\xi = (C \to S, \mathcal{M}_S, f)$ and $\xi' = (C' \to S, \mathcal{M}'_S, f')$ over a scheme S. An arrow $\xi \to \xi'$ over S is a pair (ρ, θ) as in definition 2.12 such that the following diagram commutes:



Denote by $\mathcal{I}som_S(\xi, \xi')$ the sheaf over S, which for any S-scheme $T \to S$ associates the set $Isom_T(\xi_T, \xi'_T)$ of isomorphisms of ξ_T and ξ'_T over T, where ξ_T and ξ'_T are the pull-back of ξ and ξ' via $T \to S$ respectively. Denote by $\mathcal{A}ut_S(\xi)$ the sheaf of automorphisms of ξ over S.

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