Logarithmic Geometry and Moduli

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ABSTRACT. We discuss the role played by logarithmic structures in the theory of moduli.

Contents

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1. Introduction

intro

1.1. Logarithmic structures in algebraic geometry

It can be said that Logarithmic Geometry is concerned with a method of finding and using "hidden smoothness" in singular varieties. The original insight comes from consideration of de Rham cohomology. Since singular varieties naturally occur "at the boundary" of many moduli problems, Logarithmic Geometry was soon applied in the theory of moduli.

why caps?

The main body of work on logarithmic geometry has been concerned with deep applications in the cohomological study of p-adic and arithmetic schemes. This gave the theory an aura of "yet another extremely complicated theory", in the same general baggage as Fontaine's big rings. It is a bit unfortunate that the founders were not inclined to pursue simple geometric applications right away to scatter this aura. The treatments of the the theory are however quite accessible. We hope to convince the reader here that the theory is simple enough and useful

reword.

omit?

enough to be considered by anybody interested in moduli of singular varieties, indeed enough to be included in a Handbook of Moduli.

1.2. Normal crossings and logarithmic smoothness

So what is the original insight? Let X be a nonsingular complex variety, S a curve with a point s and $f : X \to S$ a dominant morphism smooth away from s, in such a way that $f^{-1}s = X_s = Y_1 \cup \ldots \cup Y_m$ is a reduced simple normal crossings divisor. Then of course $\Omega_{X/Y} = \Omega_X / f^*\Omega_Y$ fails to be locally free at the singular points of f. But consider instead the sheaves $\Omega_X(\log(X_s))$ of differential forms with at most logarithmic poles along the Y_i, and similarly $\Omega_S(\log(s))$. Then there is an injective sheaf homomorphism $f^*\Omega_S(\log(s)) \to \Omega_X(\log(X_s))$, and *the quotient sheaf* $\Omega_f^{\log}$ *is locally free.*

So in terms of logarithmic forms, *the morphism f is as good as a smooth morphism.*

There is much more to be said: first, this $\Omega_f^{\log}$ can be extended to a logarithmic de Rham complex, and its hypercohomology, while not recovering the cohomology of the singular fibers, does give rise to the limiting Hodge structure. So it is evidently worth considering.

Second, the picture is quite a bit more general, and can be applied to all toric and toroidal maps between toric varieties or toroidal embeddings (with a little caveat about the characteristic of the residue fields). So there is some flexibility in choosing $X \to S$.

1.3. The search for a structure

Since we are considering moduli, then as soon as we consider $X \to S$ as above we must also consider the normal crossings fiber $X_s \to \{s\}$. But what structure should we put on this variety? The notion of differentials with logarithmic poles along X_s is not in itself intrinsic to X_s. Also the normal crossings variety X_s is not in itself toric or toridal, so a new structure is needed to incorporate it into the picture.

One is tempted to consider varieties which are assembled from nice variety by some sort of gluing, as normal crossings varieties are. But already normal crossings varieties do not give a satisfactory answer in general, bacause their deformation spaces have "bad" components. Here is a classical example: consider a smooth projective variety Z such that $Pic^0(Z)$ is nontrivial. Let L be a line bundle on Z and set $Y = \mathbb{P}(\mathcal{O} \oplus L)$, with zero section $Z \subset Y$. Let X be the blowing up of $Z \times 0 \subset Y \times \mathbb{A}^1$. We have a flat morphism $f : X \to \mathbb{A}^1$ with fiber $X_0 = f^{-1}(0) \simeq Y \cup Y$, where the two copies of Y are glued with the zero section of one attached to the ∞ section of the other.

So clearly X_0 is a normal crossings variety with a nice smoothing to a copy of Y. But there are other deformations: the variety $Y \cup Y$ also deforms to $Y \cup Y'$

where $Y' = \mathbb{P}(\mathcal{O} \oplus L')$ and L' a deformation of the line bundle L. And it is not hard to see that $Y \cup Y'$ does not have a smoothing. Ideally one really does not want to see this deformation $Y \cup Y'$ in the picture - and ideally X_0 should have a natural structure whose deformation space excludes $Y \cup Y'$ automatically.

Such a structure was proposed by Friedman in [8], where the notion of D-semistable varieties was introduced. This structure is somewhat subtle, and while it solves the issue in this case, it is not quite as flexible as one could wish. As we will see in Section 5, logarithmic structures subsume D-semistability and do provide an appropriate flexibility.

d-semistability?

1.4. Organization of this chapter

The purpose of this chapter is to briefly describe logarithmic structure and to indicate where they can be useful in the study of moduli spaces. Section 2 gives the basic definitions of logarithmic structures, and section 3 discusses logarithmic differentials and log smooth deformations, which are important in considering moduli spaces.

Section 4 gives the first example where logarithmic geometry fits well with moduli spaces: the moduli space of stable curves is the moduli space of log smooth curves. The issue of D-semistability does not arise since a nodal curve is automatically D-semistable. So the theory for curves is simple. Turning to higher dimensions, Section 5 shows how D-semistability can be described within logarithmic structures.

If one is to enlarge algebraic geometry to include logarithmic structures, the task of generalizing the techniques of algebraic geometry to logarithmic structure can certainly seem daunting. In section 6 we show how to encode logarithmic structure in terms of certain algebraic stacks. This allows us to reduce various constructions to the case of algebraic stacks. (One can argue that the theory of stacks is not simple either, but at least in the theory of moduli they have come to be accepted, with some exceptions [22].)

In section 7 we make use of logarithmic stacks to describe the complexes which govern deformations and obstructions for logarithmic structures even in the non-smooth case. This comes in handy later. For instance, even when studying moduli of log-smooth schemes, the moduli spaces tend to be singular, and their cotangent complexes are a necessary ingredients in constructing virtual fundamental classes.

Section 8 describes a beautiful construction, similar to polar coordinates, in which families of complex log smooth varieties give rise canonically to families of topological manifolds. Differential geometers have used polar coordinates on nodal curves to "make space" for monodromy to act by Dehn twists. Rounding (using Ogus's terminology) is a magnificent way to generalize this.

The immediate implications of logarithmic structures for De Rham cohomology and Hodge structures is described in Section 9.

We conclude by describing three applications, where logarithmic structures serve as the proverbial "magic powder" (term suggested by Kato and Ogus) to clarify or remove unwanted behavior from moduli spaces.

Section 10 describes a number of cases where the main irreducible component of a moduli space can be separated from other "unwanted" components by sprinkling the objects with a bit of logarithmic structures.

In Section 11 we introduce twisted curves, a central object of orbifold stable maps, and show how logarithmic structures give a palatable way to construct the moduli stack of twisted curves.

Section 12 gives background for the work of B. Kim, in which Jun Li's moduli space of relative stable maps, with its obstruction theory and virtual fundamental class, is beautifully simplified using logarithmic structures.

2. Definitions and basic properties

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In the first two sections, we will introduce the basic definitions of logarithmic geometry in the sense of [18]. A good introduction would be [18] and [30].

good introductions one

2.1. Logarithmic structures

The basic definitions are as follows:

italicizing

Definition 2.1. A monoid is a commutative semi-group with a unit. A morphism of monoids is required to preserve the unit element. We use Mon to denote the category of Monoids.

Definition 2.2. Let X be a scheme. A pre-logarithmic structure on X is a sheaf of monoids $\mathcal{M}$ on the étale site $X_{\acute{e}t}$ combined with a morphism of sheaves of monoids: $\alpha : \mathcal{M} \longrightarrow \mathcal{O}_X$, called the structure morphism, where we view $\mathcal{O}_X$ as a monoid under multiplication. A pre-log structure is called a log structure if $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$ via α. We call the pair $(X, \mathcal{M})$ a log scheme.

italics

Definition 2.3. Given two pre-log structures $\mathcal{M}$ and $\mathcal{N}$ on X. A morphism between them is a morphism $\mathcal{M} \to \mathcal{N}$ of sheaves of monoids which is compatible with the structure morphisms.

wording.

How should one think of such a beast? There are two extreme cases:

(1) If an element $m \in M$ has $\alpha(m) = x \neq 0$, one often thinks of m as some sort of partial data of a "branch of the logarithm of x". Evidently no data is added if x is invertible, but some is added otherwise. In particular, we will see later that m permits us to take the logarithmic differential dx/x of x.

(2) If $\alpha(m) = 0$ it is often the case that it m comes by restricting the log structure of an ambient space, and serves as the "ghost" of a logarithmic cotangent vector coming from that space. So the log structure "remembers" deformations that are lost when looking at the underlying scheme.

2.2. The log structure associated to a pre-log structure

Given a pre-log structure $\alpha : \mathcal{M} \to \mathcal{O}_X$ on X, we can associate a log structure $\mathcal{M}^a$ to be the push-out of

wording.

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^*) & \longrightarrow & \mathcal{M} \\ \downarrow & & \\ \mathcal{O}_X^* & & \end{array}$$

Better to say first then mention construction

in the category of sheaves of monoids on $X_{\acute{e}t}$, endowed with

$$\mathcal{M}^a \to \mathcal{O}_X \quad (a, b) \mapsto \alpha(a)b \ (a \in \mathcal{M}, b \in \mathcal{O}_X^*).$$

In this way, we obtain a functor a : (pre-log structures on X) $\to$ (log structures on X). From the universal property of push-out, any morphism of pre-log structure from a pre-log structure $\mathcal{M}$ to a log structure on X factor through $\mathcal{M}^a$ uniquely.

On the other hand we have a natural inclusion i : (log structure on X) $\hookrightarrow$ (pre-log structure on X) by viewing log structure as a pre-log structure. So we have the following lemma.

Lemma 2.4. [30, 1.1.5] *The functor a is left adjoint to i.*

Example 2.5. The category of log structures on X has an initial object, called the trivial log structure, given by the inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$. And it also has a final object, given by the identity map $\mathcal{O}_X \to \mathcal{O}_X$.

NClog *Example* 2.6. Let X be a regular scheme, $D \subset X$ is a normal crossing divisor. We can define a log structure $\mathcal{M}$ on X associated to the divisor D as

$$\mathcal{M} = \{g \in \mathcal{O} : g \text{ is invertible outside } D\} \subset \mathcal{O}_X.$$

though ok, would be better to say $\mathcal{M}(U) = \{g \in \mathcal{O}_X(U) : g|_{U-(D\cap U)} \in \mathcal{O}_X^*\}$

wording change

Of course this can be done for any dense open subset. But the normal crossings situation is special - we will see later taht it is *log smooth.*

Note that the concept of normal crossing is local in the étale topology. This is one reason we use the étale topology instead of the Zariski topology.

Afflog *Example* 2.7. Let P be a monoid, R a ring. Denote X Spec $R[P]$, then X has a canonical log structure associated to the canonical map $P \to R[P]$. We usually use Spec$(P \to R[P])$ to denote the log scheme X with its canonical log structure.

2.3. The inverse image

Let $f : X \to Y$ be a morphism of schemes. Given a log structure $\mathcal{M}_Y$ on Y, we can obtain a log structure on X called the invese image of $\mathcal{M}_Y$, to be the log structure associated to the pre-log structure $f^{-1}(\mathcal{M}_Y) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X$. This is usually denoted by $f^*(\mathcal{M}_Y)$. Using the inverse image of log structures, we can give the following definition.

should say somewhere that R[P] denotes the monoid algebra

Definition 2.8. A morphism of log schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ consists of a morphism of underlying schemes $f : X \to Y$, and a morphism $f^\flat : f^*\mathcal{M}_Y \to \mathcal{M}_X$ of log structures on X.

We use *LSch* to denote the category of log schemes.

Example 2.9. In Example 2.7, the log structure on $\mathrm{Spec}(P \to R[P])$ can be viewed as the inverse image of the log structure on $\mathrm{Spec}(P \to \mathbb{Z}[P])$ via the canonical map $\mathrm{Spec}(R[P]) \to \mathrm{Spec}(\mathbb{Z}[P])$.

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Example 2.10. Let k be a field, Y=Spec $k[x_1, \cdots, x_n]$, D=$V(x_1 \cdots x_r)$. Note that D is a normal crossing divisor in Y. By example 2.6, we have a log structure $\mathcal{M}_Y$ on Y associated to the divisor D. In fact, $\mathcal{M}_Y$ can be viewed as a subsheaf of $\mathcal{O}_Y$ generated by $\mathcal{O}_Y^*$ and $\{x_1, \cdots, x_r\}$.

Consider the inclusion $j : p = \mathrm{Spec}\ k \hookrightarrow Y$ sending the point to the origin of Y. Then $j^*\mathcal{M}_Y = k^* \oplus \mathbb{N}^r$, and the structure map $j^*\mathcal{M} \longrightarrow \mathcal{O}_X$ is given by $(a, n_1, \cdots, n_r) \mapsto a \cdot 0^{n_1 + \cdots + n_r}$, where we assume $0^0 = 1$ and $0^n = 0$ if $n \neq 0$. Such point with the log structure above is call a logarithmic point, when $r = 1$ we call it the standard logarithmic point.

2.4. Charts of log structures

Definition 2.11. Let $(X, \mathcal{M}_X)$ be a log scheme, and P a monoid. A chart of $\mathcal{M}_X$ is a morphism $P \to \Gamma(X, \mathcal{M}_X)$, such that the associated log structure to the pre-log structure $P \to \Gamma(X, \mathcal{M}_X) \to \Gamma(X, \mathcal{O}_X)$ is $\mathcal{M}_X$. via the induced map.

In fact, a chart of $\mathcal{M}_X$ is equivalent to a morphism $f : (X, \mathcal{M}_X) \to \mathrm{Spec}(P \to \mathbb{Z}[P])$, such that $f^\flat$ is an isomorphism. In general, we have the following:

Lemma 2.12. [30, 1.1.9]

$$Hom_{LSch}((X, \mathcal{M}_X), \mathrm{Spec}(P \to \mathbb{Z}[P])) \cong Hom_{Mon}(P, \Gamma(X, \mathcal{M}_X)).$$

say what the map is.

We can also consider the charts of log morphisms.

Definition 2.13. Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of log schemes. A chart ~~of~~ for f is a triple $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ where P_X and Q_Y are the constant sheaves associated to the monoids P and Q, which satisfy the following conditions:

1. $P_X \to \mathcal{M}_X$ and $Q_Y \to \mathcal{M}_Y$ are charts of $\mathcal{M}_X$ and $\mathcal{M}_Y$;
2. the morphism of monoids $Q \to P$ makes the following diagram commutative:

$$\begin{tikzcd} Q_X \arrow[r] \arrow[d] & P_X \arrow[d] \\ f^*\mathcal{M}_Y \arrow[r] & \mathcal{M}_X. \end{tikzcd}$$

2.5. Fine log ~~structures~~ arbitrary

In general, log structures are too wild to manipulate. Next we will introduce some well-behaved log structures. Given a monoid P, we can associat a group

$$P^{gp} := \{(a,b) | (a,b) \sim (c,d) \text{ if } \exists s \in P \text{ such that } s + a + d = s + b + c\}.$$

Note that any morphism from P to an abelian group factor through P^{gp} uniquely.

Definition 2.14. P is called integral if $P \to P^{gp}$ is injective. It is called saturated if it is integral and for any $p \in P^{gp}$, if $n \cdot p \in P$ for some positive integer n then $p \in P$.

Definition 2.15. A log scheme $(X, \mathcal{M}_X)$ is said to be fine, if étale locally there is a chart $P \to \mathcal{M}$ with P a finitely generated integral monoid. If moreover P is saturated, $(X, \mathcal{M}_X)$ is called fine and saturated, and we denote it fs.

fine and $\forall \bar{x} \to X$, $\overline{\mathcal{M}}_{X,\bar{x}}$ is saturated

In the following, we will focus on fine log schemes.

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3. Differentials, smoothness, and log smooth deformations

3.1. Logarithmic differentials

In [11] Grothendieck defines a derivation as the difference of infinitesimal liftings of a section. We can do the same thing with logarithmic schemes. First, we need a concept of infinitesimal extension, which requires the following definition.

Definition 3.1. A morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is called strict if $f^\flat : f^*\mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism. It is called a strict closed immersion [1] if it is strict and the underlying map $X \to Y$ is a closed immersion in the usual sense.

Let us consider ~~the following~~ a commutative diagram: of solid arrows

$$\begin{tikzcd}
(T_0, \mathcal{M}_{T_0}) \arrow[r, "\phi"] \arrow[d, "J", "j"'] & (X, \mathcal{M}_X) \arrow[d, "f"] \\
(T_1, \mathcal{M}_{T_1}) \arrow[r, "\psi"'] \arrow[ur, dotted, shift left, "g_1"] \arrow[ur, dotted, shift right, "g_2"'] & (Y, \mathcal{M}_Y)
\end{tikzcd}$$

where j is an exact closed immersion defined by an ideal J with $J^2 = 0$. Note that T_0 and T_1 have the same underlying topological space. Then we have the following commutative diagram of sheaves of algebras given by the underlying maps:

$$\begin{tikzcd}
\mathcal{O}_{T_0} & \phi^{-1}\mathcal{O}_X \arrow[l] \arrow[dl, dotted, shift right, "g_1^\#"'] \arrow[dl, dotted, shift left, "g_2^\#"] \\
\mathcal{O}_{T_1} \arrow[u] & \psi^{-1}\mathcal{O}_Y \arrow[l] \arrow[u]
\end{tikzcd}$$

[1] In [18], this is called an exact closed immersion. But I feel more explicit if using the word "strict" instead of "exact".

Then $g_1^\# - g_2^\#$ is a derivation $\partial_{g_1-g_2} : \phi^{-1}\mathcal{O}_X \to J$ in the usual sense. We also have a commutative diagram given by the log structures:

$$\begin{tikzcd} \mathcal{M}_{T_0} & \phi^{-1}\mathcal{M}_X \arrow[l] \arrow[dl, dotted, shift left, "g_1^\flat"'] \arrow[dl, dotted, shift right, "g_2^\flat"] \\ \mathcal{M}_{T_1} \arrow[u] & \psi^{-1}\mathcal{M}_Y \arrow[l] \arrow[u] \end{tikzcd}$$

monoids do not form an abelian category. write out.

Note that we have an exact sequence of mutiplicative monoids

$$1 \to (1+J) \to \mathcal{M}_{T_1} \to \mathcal{M}_{T_0} \to 1.$$

Hence we obtain a morphism $D_{g_1-g_2} : \phi^{-1}\mathcal{M}_X \to J$ such that $\forall m \in \phi^{-1}\mathcal{M}_X$, we have $(g_1^\flat - g_2^\flat)(m) = 1 + D_{g_1-g_2}m$. Since J is now viewed as an additive group, we obtain $D_{g_1-g_2} : \phi^{-1}\mathcal{M}_X^{gp} \to J$, and it is not hard to check that $D_{g_1-g_2}(m \cdot n) = D_{g_1-g_2}(m) + D_{g_1-g_2}(n)$ for any $m, n \in \phi^{-1}(\mathcal{M}_X)$. By the definition of log structures, we also have

(1) $\alpha(m)D_{g_1-g_2}m = \partial_{g_1-g_2}(\alpha(m))$, $\forall m \in \phi^{-1}\mathcal{M}_X$;
(2) $D_{g_1-g_2}|_{\psi^{-1}\mathcal{M}_Y} = 0$.

Remark 3.2. (1) Since the log structure contains all the invertible elements in the strcture sheaf, the map $D_{g_1-g_2}$ determines $\partial_{g_1-g_2}$.
(2) The above properties show that $D_{g_1-g_2}$ behaves like "$d\log$". This is one of the reasons for the name "logarithmic structure".

Summarizing the above discussion gives the following definitions:

first define what is a log derivation then define the sheaf.

LogDer *Definition* 3.3. Let $\overline{X} = (X, \mathcal{M}_X)$, $\overline{Y} = (Y, \mathcal{M}_Y)$ be fine log schemes, and $f : \overline{X} \to \overline{Y}$ a morphism of log schemes. Let I be an $\mathcal{O}_X$–module. The sheaf $Der_{\overline{Y}}(\overline{X}, I)$ of log derivations of $\overline{X}$ over $\overline{Y}$ to I is the sheaf of germs of pairs (∂, D) where $\partial \in Der_Y(X, I)$ and $D : \mathcal{M}_X \to I$ such that the following conditions hold:

(1) $D(ab) = D(a) + D(b)$ for $a, b \in \mathcal{M}_X$;
(2) $\alpha(a)D(a) = \partial(\alpha(a))$, for $a \in \mathcal{M}_X$.
(3) $D(a) = 0$, for $a \in f^{-1}\mathcal{M}_Y$.

LodDiff *Definition* 3.4. Using the notation as above, we define the $\mathcal{O}_X$-module Ω_f^{log} to be the quotient $\Omega_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{gp})/\mathcal{K}$, where $\mathcal{K}$ is a $\mathcal{O}_X$-module generated by local sections of the following forms:

(1) $(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$ with $a \in \mathcal{M}_X$;
(2) $(0, 1 \otimes a)$ with $a \in Im(f^{-1}(\mathcal{M}_Y) \to \mathcal{M}_X)$.

The sheaf Ω_f^{log} is called the sheaf of logarithmic differentials.

better to say that ∃ universal log derivation, then give construction.

Remark 3.5. If we consider only fine log structures, and assume that Y is locally noetherian and X locally of finite type over Y, then $Der_{\overline{Y}}(\overline{X}, I)$ and Ω_f^{log} in the definitions above are coherent sheaves. The proof of this can be find in [30, IV.1.1]

found

We have the following universal property relating the two definitions:

Proposition 3.6. [30, IV.1.1.6] *Using the natations above, we have a natural isomorphism*

$$\mathcal{H}om_{\mathcal{O}_X}(\Omega_f^{log}, I) \cong Der_{\overline{Y}}(\overline{X}, I),$$

given by $u \mapsto (u \circ \partial, u \circ D)$, where (∂, D) are the universal derivation defined by $\partial : \mathcal{O}_X \to \Omega_{X/Y} \to \Omega_f^{log}$ and $D : \mathcal{M}_X \to \mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{gp} \to \Omega_f^{log}$.

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Example 3.7. Let $R = k[x_1, \cdots, x_n]/(x_1 \cdots x_r)$, where k is a field. Denote $X =$ Spec$R$, and $\mathcal{M}_X$ to be the log structure on X given by $\mathbb{N}^r \to R$, $e_i \mapsto x_i$, where e_i is the standard generator of the monoid $\mathbb{N}^r$. Let $(Y, \mathcal{M}_Y)$ =Spec($\mathbb{N} \to k$) be the logarithmic point described in 2.10. Now we can define a morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ by the following diagram:

space

$$\begin{tikzcd} \mathbb{N}^r \arrow[r] & R \\ \mathbb{N} \arrow[r] \arrow[u, "\Delta"] & k \arrow[u] \end{tikzcd}$$

where $\Delta : e \mapsto e_1 + \cdots + e_r$, and e is the standard generator of $\mathbb{N}$. Then it is easy to see that $Der_{(Y, \mathcal{M}_Y)}((X, \mathcal{M}_X), \mathcal{O}_X)$ is a free $\mathcal{O}_X$-module generated by $x_1 \frac{\partial}{\partial x_1}, \cdots, x_r \frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_{r+1}}, \cdots, \frac{\partial}{\partial x_n}$, with a relation $x_1 \frac{\partial}{\partial x_1} + \cdots + x_r \frac{\partial}{\partial x_r} = 0$. The sheaf Ω_f^{log} is a free $\mathcal{O}_X$-module generated by the logarithmic differentials: $\frac{dx_1}{x_1}, \cdots, \frac{dx_r}{x_r}, x_{r+1}, \cdots, x_n$, with a relation $\frac{dx_1}{x_1} + \cdots + \frac{dx_r}{x_r} = 0$.

Example 3.8. Let $h : Q \to P$ be a morphism of fine monoids. Denote $(X, \mathcal{M}_X) =$ Spec($P \to \mathbb{Z}[P]$) and $(Y, \mathcal{M}_Y) =$ Spec($Q \to \mathbb{Z}[Q]$). Then we have a morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ induced by h. A direct calculation shows that $\Omega_f^{log} = \mathcal{O}_X \otimes \mathbf{Cok}(h^{gp})$.

3.2. Logarithmic Smoothness

Let us go back to the following diagram:

$$\begin{tikzcd} (T_0, \mathcal{M}_{T_0}) \arrow[r, "\phi"] \arrow[d, "J", "j"'] & (X, \mathcal{M}_X) \arrow[d, "f"] \\ (T_1, \mathcal{M}_{T_1}) \arrow[r, "\psi"'] & (Y, \mathcal{M}_Y) \end{tikzcd}$$

(resp. étale)

where j is an exact closed immersion defined by J with $J^2 = 0$. As in the usual case, we can define log smoothness by the existence of lifting sections.

Definition 3.9. A morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of fine log schemes is called smooth (étale) if the underlying morphism $X \to Y$ is locally of finite presentation and for any commutative diagram as above, there exists étale locally on T_1 a (unique) morphism $g : (T_1, \mathcal{M}_{T_1}) \to (X, \mathcal{M}_{T_1})$ such that $\phi = g \circ j$ and $\psi = f \circ g$.

give number.

We have the following useful criterion for smoothness from [18, Theorem 3.5].

KatoStrThm

Theorem 3.10. *(K.Kato) Let $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ be a morphism of fine log schemes. Assume we have a chart $Q \to \mathcal{M}_Y$, where Q is a finitely generated integral monoid. Then the following are equivalent:*

(1) *f is log smooth(étale);*
(2) *étale locally on X, there exists a chart $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ extending the chart $Q_Y \to \mathcal{M}_Y$, satisfying the following properties.*
   (a) *The kernel and the torsion part of the cokernel (the kernel and the cokernel) of $Q^{gp} \to P^{gp}$ are finite groups of order invertible on X.*
   (b) *The induced morphism from $X \to Y \times_{Spec\ \mathbb{Z}[Q]} Spec\ \mathbb{Z}[P]$ is étale in the classical sense.*

Remark 3.11. (1) We can require $Q^{gp} \to P^{gp}$ in (a) to be injective, and replace the étaleness of $X \to Y \times_{Spec\ \mathbb{Z}[Q]} Spec\ \mathbb{Z}[P]$ in (b) by the smoothness without changing the conclusion of the theorem 3.10.

(2) The arrow in (b) shows that a log smooth ~~arrow~~ is "locally toric" relative to the base. If we consider the case $Y = \text{Spec } \mathbb{C}$ with the trivial log structure, and $X = \text{Spec}(P \to \mathbb{C}[P])$ where P is a fine, saturated and torsion free monoid. Then X is a toric variety with the action of Spec $\mathbb{C}[P^{gp}]$. According to the theorem, X is log smooth relative to Y, though the underlying space might be singular. These singularities are called toric singularities in the sense of [19]. morphism

Example 3.12. Using the theorem, we can check directly that the morphism f in example 3.7 is log smooth, but the underlying map has normal crossing singularities. We will see later that one of the major advantages of log structure is to deal with the normal crossing singularites.

For log smooth morphisms, the following proposition shows that log differentials behave like the usual differentials of smooth morphisms.

Proposition 3.13. *Let $(X, \mathcal{M}_X) \xrightarrow{f} (Y, \mathcal{M}_Y) \xrightarrow{g} (Z, \mathcal{M}_Z)$ be morphisms of fine log schemes.*

(1) *There exists an exact sequence $f^*\Omega_g^{log} \to \Omega_{g \circ f}^{log} \to \Omega_f^{log} \to 0$.*
(2) *If f is log smooth, then Ω_f^{log} is a locally free $\mathcal{O}_X$-module, and we have the following exact sequence: $0 \to f^*\Omega_g^{log} \to \Omega_{g \circ f}^{log} \to \Omega_f^{log} \to 0$.*
(3) *If $g \circ f$ is log smooth and the sequence in (2) is exact and splits locally, then f is log smooth.*

A proof can be found in [30, Chapter IV].

3.3. Logarithmic smooth deformation

After we have the log smoothness, a natural thing to do is to develop the log smooth deformation. In many cases, we would require this to be a flat deformation

Better to describe maps. i.e. Given

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

$\rightsquigarrow g^* \Omega_{X/Y} \to \Omega_{X'/Y'}$

for the underlying space. Unfortunately log smoothness does not imply flatness, so we need the following definition.

Definition 3.14. A map of fine monoids $h : Q \to P$ is called integral if the induced map on monoid algebra $\mathbb{Z}[Q] \to \mathbb{Z}[P]$ is flat.

Definition 3.15. A morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of integral log schemes is called integral if for every geometric point $\overline{x} \in X$, the map of monoids $h : f^{-1}(\mathcal{M}_Y/\mathcal{O}_Y^*)_{\overline{x}} \to (\mathcal{M}_X/\mathcal{O}_X^*)_{\overline{x}}$ is integral.

the notation $\overline{M}_X$ should probably be introduced early on.

Remark 3.16. (1) If f is integral, then étale locally we have a chart $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \xrightarrow{h} P)$ such that h is integral.
(2) If $h : Q \to P$ is integral of integral monoids, then for any integral monoid Q', the push-out of $P \leftarrow Q \to Q'$ in the category of monoids is integral. Thus integral morphisms are stable under base change by integral log schemes.
(3) Given a morphism $h : Q \to P$ of integral ~~monoids,~~ there is an explicit criterion, which looks comlicated, but sometimes is ~~useful~~ for checking integrality of h directly: if $a_1, a_2 \in Q$, $b_1, b_2 \in P$ and $h(a_1)b_1 = h(a_2)b_2$, then there exist $a_3, a_4 \in Q$ and $b \in P$ such that $b_1 = h(a_3)b, b_2 = h(a_4)b$ and $a_1a_3 = a_2a_4$.

mention that this is essentially equational criterion for flatness

Now we have the following fact from [18, 4.5].

Proposition 3.17. *If f is a log smooth and integral morphism of fine log schemes, then $\underline{f}$ the underlying map is flat in the usually sense.*

Now let us consider the following deformation problem. Given a log smooth integral morphism $f_0 : (X_0, \mathcal{M}_{X_0}) \to (B_0, \mathcal{M}_{B_0})$ of fine log schemes, and a exact closed immersion $j : (B_0, \mathcal{M}_{B_0}) \to (B, \mathcal{M}_B)$ defined by an ideal J with $J^2 = 0$. We want to find a log smooth lifting $f : (X, \mathcal{M}_X) \to (B, \mathcal{M}_B)$ ~~fits~~ into the following cartesian diagram: an

$$\begin{array}{ccc} (X_0, \mathcal{M}_{X_0}) & \hookrightarrow & (X, \mathcal{M}_X) \\ \downarrow & & \downarrow \\ (B_0, \mathcal{M}_{B_0}) & \hookrightarrow & (B, \mathcal{M}_B). \end{array}$$

fitting

earlier this was called strict closed immersion

Remark 3.18. Since f_0 is integral, and $\mathcal{M}_X/(1+J) \cong \mathcal{M}_{X_0}$, it is not hard to show that the lifting f is automatically integral and hence flat.

wording

We denote $T_{f_0}^{log} = (\Omega_{f_0}^{log})^\vee$ to be the log tangent sheaf for any morphism f_0 of fine log schemes. We have the following theorem for log smooth deformations.

this is the defn of log tangent sheaf? = Der.

Theorem 3.19. *With the notation as above, we have:*
(1) *There is a canonical obstruction $\eta \in H^2(X_0, T_{f_0}^{log} \otimes J)$ such that $\eta = 0$ if and only if there ~~is an~~ a log smooth lifting.*

exists

(2) *If $\eta = 0$, then the set of log smooth deformations form a torsor under $H^1(X_0, T_{f_0}^{log} \otimes J)$.*

(3) *The automorphism of any deformation is isomorphic to $H^0(X_0, T_{f_0}^{log} \otimes J)$.*

give ref to SGA1.

The theorem can be proved similar to the case of usual deformation theory. Another proof using logarithmic cotangent complex can be found in [35, Thm 5.6], which we will discuss later.

Satriano

4. Log smooth curves and their moduli

In this section we discuss F. Kato's paper [17] in which he gives a log geometry theoretic construction of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$ of curves. He states in the introduction a motiving philosophy which relates log geometry to compactifications of moduli spaces:

Is this a quotation? If so precise ref..

Philosophy. Since log smoothness already incorporates degenerate objects, one should expect that a moduli space of log smooth objects is already compact, and so provides a compactification of the moduli of objects where the log structure is trivial.

Along the lines of this philosophy, to compactify $\mathcal{M}_{g,n}$, we want to introduce a notion of *log curve* which extends the notion of smooth curve. We do so after some preliminaries.

following F. Kato.

4.1. Relative characteristic sheaves

def:char

Definition 4.1. Given a log scheme $(X, \mathcal{M}_X)$, the *characteristic* $\bar{\mathcal{M}}_X$ is defined as $\mathcal{M}_X/\alpha(\mathcal{O}_X^*)$. Given a morphism $(f, h) : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of log schemes, the *relative characteristic* $\bar{\mathcal{M}}_{X/Y}$ is defined as $\mathcal{M}_X/\operatorname{im}(f^*\mathcal{M}_Y \to \mathcal{M}_X)$.

\overline{\mathcal M}

ex:char

Example 4.2. Let $X = \operatorname{Spec} k[\mathbb{N}^n]$ with its canonical log structure. Given $t \in X$, let $d(t)$ be the number of coordinate hyperplanes on which t lies. Then

$$\bar{\mathcal{M}}_{X,\bar{t}} = \bar{\mathcal{M}}_{X,t} = \mathbb{N}^{d(t)},$$

where $\bar{t}$ is any geometric point mapping to t. More generally, if X is a smooth scheme and D is a normal crossing divisor on X, then we obtain a log structrue $\mathcal{M}_X$ defined by D. Then

$$\bar{\mathcal{M}}_{X,\bar{t}} = \mathbb{N}^{d(\bar{t})},$$

where $d(\bar{t})$ denotes the number of irreducible components of the inverse image of D in $\operatorname{Spec} \mathcal{O}_{X,\bar{t}}$.

should be introduced earlier.

The relative characteristic, as one might imagine, tells how much log structure one log scheme has relative to another:

Is this a meaningful statement?

Lemma 4.3 ([17, Lemma 1.6]). *If $(f, h) : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is an integral morphism of fine log schemes, then $\bar{\mathcal{M}}_{X/Y,\bar{x}} = 0$ if and only if (f, h) is strict in an étale neighborhood of x.*

one must be careful here about which cat the quotient is taken in: monoids, integral monoids etc.

As the following example illustrates, the integrality assumption on (f, h) is necessary.

ex:intrel

Example 4.4. Let P be the monoid on three generators x, y, and z subject to the relation $x + y = 2z$. We have an injection

$$i : P \longrightarrow \mathbb{N}^2$$

sending x to $(2, 0)$, y to $(1, 1)$, and z to $(0, 2)$. Let $X = \operatorname{Spec} k[\mathbb{N}^2]$ and $Y = \operatorname{Spec} k[P]$ with their canonical log structures. Then it follows from [18, Prop 3.4] that the morphism $(f, h) : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ induced by i is log étale, but it is not flat, and hence, by [18, Cor 4.5], not integral. It is easy to check that

$$\mathcal{M}_{X/Y,0} = \mathbb{N}^2/P \simeq \mathbb{Z}/2,$$

so $\bar{\mathcal{M}}_{X/Y,0} = 0$, but (f, h) is not strict.

We are now ready to introduce and discuss log curves.

was this terminology already introduced? should go in §2.

4.2. Log curves

def:logcurve

Definition 4.5. A *log curve* is a log smooth integral morphism $(f, h) : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ of fs log schemes such that the geometric fibers of f are reduced connected curves.

We require (f, h) to be integral so that, by [18, Cor 4.5], f is flat. The reason for the fs assumption is to avoid cusps, as the following example shows.

ex:cusp

Example 4.6. If $X = \operatorname{Spec} k[\mathbb{N} - \{1\}]$ is given its canonical log structure and $S = \operatorname{Spec} k$ is given the trivial log structure, then $(X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is log smooth and integral; however, $X = \operatorname{Spec} k[x, y]/(y^2 - x^3)$ has a cusp.

or worse singularities?

It is a remarkable fact that by endowing our curves with log structures as in Definition 4.5, this is enough to control the singularities of the curve.

thm:node

Theorem 4.7 ([17, Thm 1.3]). *If k is seperably closed and $(f, h) : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is a log curve with $S = \operatorname{Spec} k$, then X has at worst nodal singularities. Moreover, if $r_1, \ldots, r_\ell$ are the nodes of X, then there exist smooth points $s_1, \ldots, s_n$ of X such that*

field?

$$\bar{\mathcal{M}}_{X/S} = \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_\ell} \oplus \mathbb{N}_{s_1} \oplus \ldots \mathbb{N}_{s_n};$$

here M_x denotes the skyscraper sheaf for a monoid M supported at a point $x \in X$.

The reader should think of the s_i in the above theorem as marked points. So we can already see how n-pointed curves emerge naturally from the log geometry perspective.

ex:logcurve

Example 4.8. Consider the closed subscheme X of $\mathbb{P}^2_k \times_k \mathbb{A}^1_k$ defined by $xz = ty$, where t is the coordinate of $\mathbb{A}^1_k$ and x, y, and z are the coordinates of $\mathbb{P}^2_k$. Then X has a natural log structure $\mathcal{M}_X$. For example, on the locus where z is invertible,

X is given by $\operatorname{Spec} k[P_z]$ with P_z a monoid on five generators a, b, c, c', u subject to the relations $c + c' = 0$ and $a + c = b + u$; here $\mathcal{M}_X$ is given by the canonical log structure associated to P_z. Then the projection

$$(X, \mathcal{M}_X) \longrightarrow \mathbb{A}^1_k$$

is a log curve, where $\mathbb{A}^1_k$ is given the log structure defined by the divisor $t = 0$. We see that every fiber above $t \neq 0$ is isomorphic to $\mathbb{P}^1_k$ with log structure given by the divisor at 0 and ∞; the fiber above $t = 0$ is nodal. The n in Theorem 4.7 is equal to 2 for all geometric fibers.

Since our goal is to give a log geometric description of $\overline{\mathcal{M}}_{g,n}$, we would like to express the stability condition purely in terms of log geometry. The following proposition provides the key.

prop:dualizing

Proposition 4.9 ([17, Prop 1.13]). *With notation as in Theorem 4.7, there is a natural isomorphism*

$$\Omega^1_{(X,\mathcal{M}_X)/(S,\mathcal{M}_S)} \longrightarrow \omega_X(s_1 + \cdots + s_n),$$

where ω_X is the dualizing sheaf of X.

We therefore make the following definition.

def:stable

Definition 4.10. Let $(f, h) : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ be a log curve and for all geometric points $\bar{t}$ of S, let $\ell(\bar{t})$ and $n(\bar{t})$ be such that

$$\bar{\mathcal{M}}_{X_{\bar{t}}/\bar{t}} = \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_{\ell(\bar{t})}} \oplus \mathbb{N}_{s_1} \oplus \ldots \mathbb{N}_{s_{n(\bar{t})}}.$$

We say (f, h) is *of type* (g, n) if f is proper, X has genus g, and $n(\bar{t}) = n$ for all $\bar{t}$. We say (f, h) is *stable of type of* (g, n) if it of type (g, n) and

$$H^0(X_{\bar{t}}, T_{(X_{\bar{t}}, \mathcal{M}_{X_{\bar{t}}})/(\bar{t}, \mathcal{M}_{\bar{t}})}) = 0$$

for all geometric points $\bar{t}$ of S.

refer back to 3.19 and reconcile notations.

4.3. Moduli

Let $\overline{\mathcal{M}}^{log}_{g,n}$ be the stack of stable log curves of type (g, n). This requires some explanation. Namely, there are two equivalent ways we can view $\overline{\mathcal{M}}^{log}_{g,n}$: as a stack over the category of fs log schemes (with the strict étale topology), or as a stack over the category of schemes equipped with a log structure. In the former, $\overline{\mathcal{M}}^{log}_{g,n}((S, \mathcal{M}_S))$ is the category of stable log curves of type (g, n) over $(S, \mathcal{M}_S)$. In the latter, $\overline{\mathcal{M}}^{log}_{g,n}(S)$ is the category of fs log structures $\mathcal{M}_S$ on S together with a stable log curve of type (g, n) over $(S, \mathcal{M}_S)$. In this second case, we obtain an fs log structure $\mathcal{M}$ on $\overline{\mathcal{M}}^{log}_{g,n}$ defined by the following property. If $\varphi : S \to \overline{\mathcal{M}}^{log}_{g,n}$ is a morphism which corresponds to the choice $\mathcal{M}_S$ of fs log structure on S along with a stable log curve of type (g, n) over $(S, \mathcal{M}_S)$, then $\varphi^*\mathcal{M} = \mathcal{M}_S$.

This should be expanded. Probably discuss abstractly and also make clear which of the two ways you view $\mathcal{M}_{g,n}$

It is, in fact, true ([17, Prop 1.7]) that if $(f,h) : (X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ is a log curve of type (g,n), then the s_i in each geometric fiber fit together to yield n sections σ_i of f. It follows then that every stable log curve of type (g,n) is an n-pointed stable curve of genus g in the classical sense. We therefore obtain a morphism

$$F : \overline{\mathcal{M}}_{g,n}^{log} \longrightarrow \overline{\mathcal{M}}_{g,n},$$

How is $\mathcal{M}_{g,n}^{log}$ a stack over cat of schemes?

which is *a priori* only a morphism of stacks over the category of schemes. However, $\overline{\mathcal{M}}_{g,n}$ carries a log structure as well given by the divisor at infinity. The main theorem of [17] is then

Theorem 4.11 ([17, Thm 4.5]). *The morphism F is an equivalence of stacks with log structures.*

We elaborate upon the proof that F is essentially surjective. It is shown that there is a canonical way to add log structure to any n-pointed stable curve $f : X \to S$ of genus g to realize it as a stable log curve of type (g,n). Lemmas 2.1 and 2.2 of [17] show that it is enough to consider the case when $S = \operatorname{Spec} A$ and A is strict Henselian. For every node r_i of the closed fiber of f, we can find an étale neighborhood U_i of the points specializing to r_i and a diagram

$$\begin{array}{ccc} U_i & \xrightarrow{\psi_i} & \operatorname{Spec} A[x,y,t]/(xy-t) \\ \downarrow & & \downarrow{\scriptstyle \pi} \\ S & \xrightarrow{\varphi_i} & \operatorname{Spec} A[t] \end{array}$$

which is cartesian. Let $t_i \in A$ be the image of t under the morphism induced by φ_i. Endowing $\operatorname{Spec} A[t]$ with the log structure associated to the morphism $\mathbb{N} \to A[t]$ sending 1 to t, and $\operatorname{Spec} A[x,y,t]/(xy-t)$ with the log structure associated to $\mathbb{N}^2 \to A[x,y,t]/(xy-t)$ sending e_1 (resp. e_2) to x (resp. y), we see that (π, Δ) is a morphism of log schemes, where $\Delta : \mathbb{N} \to \mathbb{N}^2$ is the diagonal map. Pulling back these log structures under φ_i (resp. ψ_i), we obtain log structures $\mathcal{L}_i$ (resp. $\mathcal{M}'_i$) on S (resp. U_i). Away from the points specializing to r_i, we define a log structure $\mathcal{M}''_i$ as the pullback of $\mathcal{L}_i$. The log structures $\mathcal{M}'_i$ and $\mathcal{M}''_i$ glue to yield a log structure $\mathcal{M}_i$ on X. Let $\mathcal{N}$ be the log structure on X associated to the divisor defined by the marked points. We let

$$\mathcal{M}_X = \mathcal{M}_1 \oplus_{\mathcal{O}_X^*} \cdots \oplus_{\mathcal{O}_X^*} \mathcal{M}_\ell \oplus_{\mathcal{O}_X^*} \mathcal{N}$$

and

$$\mathcal{M}_S = \mathcal{L}_1 \oplus_{\mathcal{O}_S^*} \cdots \oplus_{\mathcal{O}_S^*} \mathcal{L}_\ell.$$

It is not difficult to see that with these definitions, we have endowed f with the structure of a log curve.

es-big-picture

4.4. Back to the big picture

We end by mentioning a type of converse to the philosophic principal mentioned at the beginning of this section. We have seen that since log smoothness includes degenerate objects, log geometry can naturally lead to compactifications; however, it is also generally true that we do not end up with "too many" degenerate objects.

Philosophy. Log geometry controls degenerations.

In higher dimensions, compactifications tend to have unwanted extra components. Log geometry helps to cut down on these components. Let us give some inkling of an idea as to why this should be true. Suppose $\mathfrak{X}$ is an algebraic stack which is irreducible. Suppose we can find a proper algebraic stack $\bar{\mathfrak{X}}$ with a fine log structure and an open immersion $i : \mathfrak{X} \to \bar{\mathfrak{X}}$ such that $\mathfrak{X}$ is the trivial locus of $\bar{\mathfrak{X}}$.

As we now explain, log geometry provides us with a good method of trying to show that $\bar{\mathfrak{X}}$ is irreducible as well. If k is separably closed and $x : \operatorname{Spec} k \to \bar{\mathfrak{X}}$ is a morphism, then pulling back the log structure on $\bar{\mathfrak{X}}$ endows $\operatorname{Spec} k$ with a fine log structure. By [18, Lemma 2.10], it follows that this is the log structure associated to a morphism of monoids $P \to k$, where P is fine. Hence, we have a strict closed immersion of log schemes $j : \operatorname{Spec} k \to \operatorname{Spec} k[[P]]$, where $\operatorname{Spec} k[[P]]$ is given its canonical log structure. Note that the generic point $\operatorname{Spec} K$ of $\operatorname{Spec} k[[P]]$ carries the trivial log structure. Therefore, if x factors as a morphism of log stacks through j, then we automatically obtain a commutative diagram

$$\begin{tikzcd}
\mathfrak{X} \arrow[rr, "i"] & & \bar{\mathfrak{X}} \\
\operatorname{Spec} K \arrow[u, "y"] \arrow[r] & \operatorname{Spec} k[[P]] \arrow[ur] & \operatorname{Spec} k \arrow[l, "j"] \arrow[u, "x"']
\end{tikzcd}$$

insert references to literature

and hence x is the specialization of the point y of $\mathfrak{X}$. We see then that the log structure on $\operatorname{Spec} k$ obtained from $\bar{\mathfrak{X}}$ somehow serves as a compass telling us which way to look in order to find a family degenerating to our given point x of $\bar{\mathfrak{X}}$.

D-SS

5. D-semistability and log structures

has this been properly defined?

Convention. Throughout this chapter, X will be a normal crossing variety over an algebraically closed field k. $x \in X$ means a closed point of X. By a standard neighborhood of $x \in X$, we mean an étale neighborhood of x of the form $\operatorname{Spec} \dfrac{k[x_1, \cdots, x_n]}{(x_1 \cdots x_r)} \to X$ such that x corresponds to the point with coordinate $(0, \cdots, 0)$. $\mathcal{M}_k$ denotes the log structure of the standard log point $\mathbb{N} \to k, n \mapsto 0^n$.

insert more motivating discussion.

5.1 Introduction

To study the geometry of a normal crossing variety X, one would like to ask the following question:

$\boxed{1}$ (1) Can we embed X into another variety with codimension 1, i.e. an embedding $i : X \to \mathscr{X}$ as a normal crossing divisor?

$\boxed{2}$ (2) Can we find a semi-stable smoothing of X, i.e. embed $X \to \operatorname{Spec} k$ in a flat family over a curve $\mathscr{X} \to C$, s.t. there exists a diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] & \mathscr{X}^* = \mathscr{X}\backslash X \arrow[l] \arrow[d, "f^*"] \\ \operatorname{Spec} k \arrow[r, "0"] & C & C^* = C\backslash 0 \arrow[l] \end{tikzcd}$$

fix numbering.

where $\mathscr{X}$ is smooth, the squares are cartesian and f^* is smooth?

The answer to these questions are not always yes, because the existence of such maps would imply the existence of certain log structures on X, which in turn would imply intrinsic condition on X, so their existence is not guaranteed.

$\boxed{\texttt{emb}}$ *Example* 5.1 (log structure of embedding type). If we can find an embedding as in 1 above, then $i^*(j_*\mathcal{O}^\times_{\mathscr{X}^*}) \to \mathcal{O}_X$ defines a log structure $\mathcal{M}_X$ on X, which étale locally has a chart $\mathbb{N}^r \to \operatorname{Spec} \dfrac{k[x_1, \cdots, x_n]}{(x_1 \cdots x_r)}$ sending the element e_i in the standard basis of $\mathbb{N}^r$ to x_i, making X a log smooth variety over $\operatorname{Spec} k$ with the trivial log structure. This is called a log structure of embedding type.

Remark. This construction is functorial in X.

$\boxed{\texttt{sm}}$ *Example* 5.2 (log structure of semi-stable type). If we can find a semi-stable smoothing as in 2 above, then by the remark on functoriality, what we have in this case is not only a log structure of embedding type on X, but also a morphism (of sheaf of monoids) $f^\flat : f^*\mathcal{M}_k \to \mathcal{M}_X$, which makes X a log smooth variety over $(\operatorname{Spec} k, \mathcal{M}_k)$. Étale locally a chart for the log structure on X can be put in the form $\left(\operatorname{Spec} \dfrac{k[x_1, \cdots, x_n]}{(x_1 \cdots x_r)}, \mathbb{N}^r, e_i \mapsto x_i, i = 1, \cdots, r\right)$. Modulo the units in the monoid, the morphsim of quotient monoids induced by $f^\flat : f^*\mathcal{M}_k \to \mathcal{M}_X$ is just the diagonal $\Delta : \mathbb{N} \to \mathbb{N}^r$. Such a pair $(\mathcal{M}_X, f^\flat : f^*\mathcal{M}_k \to \mathcal{M}_X)$ is called a log structure of semi-stable type on X.

Remark. The log structure $\mathcal{M}_k$ on $\operatorname{Spec} k$ can be defined by the pullback of the log structure $(\mathcal{M}_0 = j_*\mathcal{O}^\times_{C^*} \to \mathcal{O}_C)$ on C, i.e. the log structure defined by the divisor 0 of C. We have an isomorphism $\overline{\mathcal{M}_k} := \mathcal{M}_k/k^\times \cong (\mathcal{M}_0/\mathcal{O}^\times_C)_0 \cong \mathbb{N}$, where the second isomorphism assigns each function to its vanishing order at the point 0 in $\mathbb{N}$. This gives a geometric interpretation of the standard log point.

Concerning the existence of such log structure on the normal crossing variety X, we have:

embed **Theorem 5.3.** *Let X be a normal crossing variety over the spectrum of an algebraically closed field, then X can be equipped with a log structure of embedding type iff there exists a line bundle $\mathcal{L}$ on X such that*

$$\mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X)|_D \cong \mathcal{L}|_D$$

where D is the non-smooth locus of X.

insert precise refs.

d-semi **Theorem 5.4** (d-semistability). *Let X be a normal crossing variety over the spectrum of an algebraically closed field, then X can be equipped with a log structure over the standard log point, such that the structure morphism is log smooth iff it is d-semistable, i.e.,*

$$\mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X)|_D \cong \mathcal{O}_D$$

this should be defined separately from theorem.

where D is the non-smooth locus of X.

Remark. # It is not hard to see $\mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X)|_D$ is a line bundle on D.

Generalization of these theorems can be found in [33], section 3.

explain why this follows from 5.4.

Corollary 5.5. *If X has a smoothing, then X is d-semistable.*

~~*Remark.*~~ Being d-semistable is not equivalent to being semi-stably smoothable. See [39] for counter examples.

what does this mean?

precise ref.

Example 5.6 (a normal crossing variety that is not d-semistable[8]). Let X be the subvariety of $\boldsymbol{P}^3$ defined by the product of 4 linear equations $f = L_1L_2L_3L_4 = 0$, it is of normal crossing, provided the four planes has no points in common. Then D is defined by the homogeneous ideal $(L_iL_j|1 \le i < j \le 4)$, and it is not hard to calculate that $\mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X)|_D \cong \mathcal{O}_D(4)$, which is not trivial. So X is not d-semistable.

also discuss example from introduction

Corollary 5.7. *The X above has no semi-stable smoothings.*

refer by #

Remark. If we put X in the 1−dimensional familly $\mathscr{X}$ defined by $f + tg = 0$ with parameter t, where g is a smooth quartic, then X is the fiber over $t = 0$ and for generic g, $\mathscr{X}$ over $(t \neq 0)$ is smooth. But the whole space of this family is not smooth: in fact, for a generic g, this family is singular at the 24 points of $D \cap \{g = 0\}$. However, a single blowing up at such a point gives a $\boldsymbol{P}^1 \times \boldsymbol{P}^1$. Contract along either ruling gives back a family with parameter t which has one less singularity. If we do this process to all 24 points, we will get a family which is a semi-stable smoothing ~~of $\tilde{X}$~~, the blowing-up of X at those 24 points. $\tilde{X}$ is d-semistable. For details, see [8].

precise ref.

since we are not including proofs, this should only be included if giving more insight.

5.2. ~~Strategy of the Proof~~ Refined analysis of existence of log structures.

Let's analyze the situation. we want to:

(1) Put a log structure of embedding type on X.

(2) See if it is possible to make that log structure semi-stable.

We will see that two related obstructions arise naturally, where the vanishing of the first corresponds to the first step, and the vanishing of the second, which means precisely being d-semistable, allows us to do the second step.

Since étale locally a log structure of embedding type always exists, let's consider the stack $\mathcal{G}$, which to each $U \in X_{\acute{e}t}$ associates the groupoid of log structure of embedding type on U. Then it is not hard to show (using Artin's approximation theorem) that any two element of U is locally isomorphic, which means $\mathcal{G}$ is a gerbe. Since $\operatorname{Aut}\mathcal{G} \cong \mathcal{K}$, where $\mathcal{K}$ is the kernel of the restriction map $\mathcal{O}_X^\times \to \mathcal{O}_D^\times$, we have:

Proposition 5.8. *There is an obstruction η in $H^2(X_{\acute{e}t}, \mathcal{K})$ whose vanishing is equivalent to the existence of a log structure of embedding type on X.*

For the calculation of this obstruction, we state the following result (See [16] [2] [33]): precise ref.

gerbe

Proposition 5.9. *In the long exact sequence of cohomology associated to the short exact sequence $1 \to \mathcal{K} \to \mathcal{O}_X^\times \to \mathcal{O}_D^\times \to 1$, the line bundle $\mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X)|_D$ maps to $-\eta \in H^2(X_{\acute{e}t}, \mathcal{K})$*

Combining these results with the exactness of the long exact sequence , we have proved Theorem 5.3.

Remark. General theory tells us if $\eta = 0$, then the set of all log structure of embedding type on X is naturally a torsor under $H^1(X_{\acute{e}t}, \mathcal{K})$. In general the set of all log structure of embedding type is only a pseudo torsor under this group.

Suppose $\eta = 0$, then we can put a log structure of embedding type on X, which maps to $(\operatorname{Spec} k, k^\times)$, to make it a log structure of semistable type, we need a morphism of monoids $f^*\mathcal{M}_k \to \mathcal{M}_X$, such that in an standard neighborhood of $x \in X$, on the level of charateristic[3], is just the diagonal $\Delta : \mathbb{N} \to \mathbb{N}^r$, where r is the number of irrducible components passing through x. fix sentence

Since $\mathcal{M}_k \cong \mathbb{N} \oplus k^\times$ (non-canonically), and the image of $k^\times \subset \mathcal{M}_k$ is determined by the underlying morphism of schemes. To give the morphism wanted

[2]An early version of this paper, *Logarithmic embeddings and logarithmic semistable reductions*, http://arxiv.org/PS_cache/alg-geom/pdf/9411/9411006v2.pdf, is probably more relevant to our topic here.

[3]Here we used the terminology of F.Kato, the characteristic of a log structure $\alpha : \mathcal{M}_X \to \mathcal{O}_X$ is $\overline{\mathcal{M}_X} := \mathcal{M}_X/\alpha(\mathcal{O}_X^\times)$

from $\mathcal{M}_k$ to $\mathcal{M}_X$, we only have to specify the image of an element in $\mathcal{M}_k$ having vanishing order 1.

Now the question becomes a lifting problem for morphism of sheaves in monoids:

$$\begin{tikzcd} & \mathcal{M}_X \arrow[d, "\beta"] \\ \mathbb{N} \cong f^{-1}\overline{\mathcal{M}_k} \arrow[r, "\overline{f^\flat}"] \arrow[ur, dotted] & \overline{\mathcal{M}_X} \end{tikzcd}$$

where étale locally $\overline{f^\flat}$ is the diagonal Δ. To lift it is equivalent to lift $\Delta(1)$.

Consider the sheaf of all the possible local liftings of $\Delta(1)$, $T = \beta^{-1}(\Delta(1))$, then T is a torsor under $\mathcal{O}_X^\times$. To find a lifting of $\Delta(1)$ is equivalent to find a global section of T, i.e. a trivialization of T.

It seems like we've got an obstruction of finding a log structure on X which is semi-stable in $H^1(X_{ét}, \cancel{\mathcal{O}_X) =}\,\mathrm{Pic}(X)$. This is, however, not quite true. In fact, what we got is for each log scheme X with a log structure of embedding type, the obstructi~~on of making it semi-sta~~ble. And our original question (on the existence ~~of lo~~g structure of semi-stable type) allows some ambiguity of choosing the log structure of embedding type $\mathcal{M}_X$ on X. As we remarked after Prop.5.9, in this case the set of all log structure of embedding type on X is an $H^1(X_{ét}, \mathcal{K})$-torsor, which implies[4]:

don't understand.

Proposition 5.10. *If $\eta = 0$, i.e. there exists a log structure of embedding type on X. Then there is an obstruction of finding a log structure of semi-stable type on X, $\eta' \in H^1(X_{ét}, \mathcal{O}_X^\times)/H^1(X_{ét}, \mathcal{K})$. There is a log structure of semi-stable type iff $\eta' = 0$.*

fix wording

By the long exact sequence of cohomology, $H^1(X_{ét}, \mathcal{O}_X^\times)/H^1(X_{ét}, \mathcal{K})$ embeds into $H^1(D_{ét}, \mathcal{O}_D^\times) \cong \mathrm{Pic}(D)$. For the calculation of η' as an element of $\mathrm{Pic}(D)$, we state the following proposition (See [33]).

Proposition 5.11. *We have $-\eta' = [\mathcal{E}xt^1_X(\Omega^1_X, \mathcal{O}_X)|_D] \in \mathrm{Pic}(D)$.*

Combining these two proposition, we have proved Theorem 5.4.

LogStacks

6. Stacks of logarithmic structures

6.1. A motivating example

Before introducing the stack LOG_S classifying fine log structures on schemes over a fine log scheme S, constructed by Olsson in [32], let's look at an example, which will give the local covers of LOG_S.

~~[4]Some compatibility should be checked here, which we left to the readers.~~

For a fine log scheme X with log structure $(\mathcal{M}_X, \alpha : \mathcal{M}_X \to \mathcal{O}_X)$, we identify $\mathcal{O}_X^*$ with its inverse image under α, regarded as a subsheaf of monoids in $\mathcal{M}_X$. Let $\overline{\mathcal{M}}_X$ be the quotient sheaf $\mathcal{M}_X/\mathcal{O}_X^*$ for the étale topology, called the characteristic of X. We write $\underline{X}$ for the underlying scheme of the log scheme X. See [18].

refer to earlier

for what?

Put in def environment and fix wording.

Define the *Deligne-Faltings log structure* of rank r on $\underline{X}$ to be a family $(L_1, \cdots, L_r)$ of line bundles on $\underline{X}$ together with maps $s_i : L_i \to \mathcal{O}_X$ of line bundles, for each i.

Consider the following three functors from schemes to groupoids:

you mean fibered categories?

(1) $X \mapsto \{\text{Deligne-Faltings log structures of rank 1 on } X\}$;

(2) $X \mapsto \{\text{fine log structures } \mathcal{M}_X \xrightarrow{\alpha} \mathcal{O}_X \text{ with a morphism of sheaves of monoids } \mathbb{N} \xrightarrow{\beta} \overline{\mathcal{M}}_X \text{ that étale locally lifts to a chart: } \mathbb{N} \xrightarrow{\widetilde{\beta}} \mathcal{M}_X\}$;

(3) $X \mapsto [\mathbb{A}^1/\mathbb{G}_m](X)$, where the quotient stack is formed with respect to the multiplication action of $\mathbb{G}_m$ on $\mathbb{A}^1$.

insert ref.

Lemma 6.1. *These three functors are equivalent.*

Let's sketch the proof. Given a DF log structure $(L, s : L \to \mathcal{O}_X)$ of rank 1 on X, define a sheaf of monoids $\mathcal{M}'$ on X to be

$$\coprod_{n \geq 0} \underline{\mathrm{Isom}}(\mathcal{O}_X, L^{\otimes n}),$$

the sheafification of the presheaf that takes U to $\coprod_{n\geq 0} \mathrm{Isom}(\mathcal{O}_U, (L|_U)^{\otimes n})$. It comes with a natural morphism of sheaves of monoids $\mathcal{M}' \to \mathbb{N}$. The monoid structure on $\mathcal{M}'$ is the obvious one:

$$(n, a : \mathcal{O} \to L^{\otimes n}) \cdot (m, b : \mathcal{O} \to L^{\otimes m}) = (n + m, a \otimes b).$$

The map $s : L \to \mathcal{O}_X$ induces a morphism

$$\underline{\mathrm{Isom}}(\mathcal{O}_X, L^{\otimes n}) \xrightarrow{\otimes s} \underline{\mathrm{Hom}}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$$

of sheaves, hence giving a pre-log structure on $\mathcal{M}'$:

$$\mathcal{M}' \to \mathcal{O}_X.$$

We take $\mathcal{M}_X$ to be the log structure associated to this pre-log structure $\mathcal{M}'$. Note that $\mathcal{M}'/\mathcal{O}_X^* \cong \mathbb{N}$, and we define $\beta : \mathbb{N} \to \overline{\mathcal{M}}_X$ to be the composite

$$\beta : \mathbb{N} \cong \mathcal{M}'/\mathcal{O}_X^* \to \mathcal{M}/\mathcal{O}_X^*.$$

Locally the line bundle L is trivial, and one can choose a trivialization of L, which gives trivializations of all $L^{\otimes n}$. Sending $n \in \mathbb{N}$ to this trivialization defines a section $\mathbb{N} \to \mathcal{M}'$, and hence a section $\widetilde{\beta} : \mathbb{N} \to \mathcal{M}' \to \mathcal{M}_X$. One can check that this is a chart.

Conversely, given a fine log structure $(\mathcal{M}_X, \mathcal{M}_X \xrightarrow{\alpha} \mathcal{O}_X)$ on X with a morphism $\beta : \mathbb{N} \to \overline{\mathcal{M}}_X$ that étale locally lifts to a chart $\widetilde{\beta} : \mathbb{N} \to \mathcal{M}_X$, we have a section $\beta(1)$ of $\overline{\mathcal{M}}_X$, and its inverse image under $\pi : \mathcal{M}_X \to \overline{\mathcal{M}}_X$ is an $\mathcal{O}_X^*$-torsor, which corresponds to a line bundle L. The composition

$$\pi^{-1}(\beta(1)) \subset \mathcal{M}_X \xrightarrow{\alpha} \mathcal{O}_X$$

gives a morphism of line bundles $s : L \to \mathcal{O}_X$.

Giving a morphism $X \to [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to giving a $\mathbb{G}_m$-torsor (namely a line bundle L) with a $\mathbb{G}_m$-equivariant morphism to $\mathbb{A}^1$:

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

This diagram is equivalent to the following one

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{A}^1_X \\ \downarrow & & \downarrow \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]_X, \end{array}$$

since $[\mathbb{A}^r/\mathbb{G}_m^r] \simeq [\mathbb{A}^1/\mathbb{G}_m]^{\times r}$

and the top arrow induces a $\mathbb{G}_m$-equivariant morphism $Y \to \mathbb{A}^1_X$, namely a morphism of line bundles $s : L \to \mathcal{O}_X$. This finishes the proof.

In fact, in the three functors, one can replace $\mathbb{N}$ by $\mathbb{N}^r$, and rank 1 DF-log structure by rank r DF-log structure, and replace $[\mathbb{A}^1/\mathbb{G}_m]$ by $[\mathbb{A}^r/\mathbb{G}_m^r]$, and they are still equivalent.

More generally, let P be a fine monoid and S a scheme, and let $S[P]$ be the product $S \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}[P]$, which has a fine log structure coming from the chart $P \to \mathbb{Z}[P]$. Let P^{gp} be the associated group, then we have the following.

insert ref.

Lemma 6.2 *The following two functors from S-schemes to groupoids are equivalent:*

(1) $X \mapsto \{$*fine log structures* $\mathcal{M}_X \xrightarrow{\alpha} \mathcal{O}_X$ *with a morphism of sheaves of monoids* $P \xrightarrow{\beta} \overline{\mathcal{M}}_X$ *that fppf locally lifts to a chart:* $P \xrightarrow{\tilde{\beta}} \mathcal{M}_X\}$;

(2) $X \mapsto [S[P]/S[P^{gp}](X)$.

If in addition P is fs, then one can replace "fppf" by "étale".

Does this need more explanation?

The S-group scheme $S[P^{\text{gp}}]$ acts on $S[P]$ by translation. Note that for an affine S-scheme Spec R, the set of R-points $S[P](R)$ is the set of monoid homomorphisms $\operatorname{Hom}_{\text{mon}}(P, R)$, where R is regarded as a multiplicative monoid. When $S = \operatorname{Spec} k$ for a field k and P is saturated and torsion-free, the variety $S[P]$ with this action of the torus $S[P^{\text{gp}}]$ is a toric variety, and the stack quotient $\mathcal{S}_P = [S[P]/S[P^{\text{gp}}]]$ is a toric stack.

defn?

6.2. The stack of log structures

Now we can discuss the stack LOG_S parameterizing fine log structures.

Let S be a fine log scheme. Define LOG_S to be the category with

- objects: morphisms $(X, \mathcal{M}_X) \to (S, \mathcal{M}_S)$ of fine log schemes, and
- morphisms: strict morphisms $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ over S.

With the functor $(X \to S) \mapsto (\underline{X} \to \underline{S})$ from $LOG_S \to \mathrm{Sch}_{\underline{S}}$, this defines a fibered category over $\underline{S}$. One of the main results in [32] is the following.

Theorem 6.3. *LOG_S is an algebraic stack locally of finite presentation over $\underline{S}$.*

Here for an algebraic stack we use a slightly different definition from [9, 4.1]. Namely the first axiom there that the diagonal is representable, separated and quasi-compact, is replaced by that the diagonal is representable and of finite presentation. In fact the stack LOG_S is not quasi-separated [32, 3.17].

Here are two basic properties of LOG_S.

Proposition 6.4. *(1). The natural map $i_S : \underline{S} \to LOG_S$ corresponding to the given fine log structure $\mathcal{M}_S$ on S is an open immersion;*

(2). The 2-functor id: $S \longrightarrow S$

$$S \mapsto LOG_S : \{\textit{fine log schemes}\} \to \{\textit{algebraic stacks}\}$$

preserves fiber product. More precisely, if

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

is a Cartesian square of fine log schemes, then the induced diagram

$$\begin{array}{ccc} LOG_{X'} & \longrightarrow & LOG_X \\ \downarrow & & \downarrow \\ LOG_{S'} & \longrightarrow & LOG_S \end{array}$$

(2) is immediate?

is a 2-Cartesian square of algebraic stacks.

We don't give their proof here, which can be found in [32]. precise ref.

6.3. What is LOG_S good for?

One can use this stack LOG_S to reinterprete many concepts in log geometry. Note that for a morphism $f : X \to S$ of fine log schemes, the induced morphism $LOG(f) : LOG_X \to LOG_S$ is faithful, hence representable.

Definition 6.5. Let P be a property of representable morphisms of algebraic stacks. Then we say that $f : X \to S$ has property $LOG(P)$ if $LOG(f) : LOG_X \to LOG_S$

has property P. We say that f has property weak $LOG(P)$ if the map $\underline{X} \to LOG_S$ corresponding to the given log structure $\mathcal{M}_X$ has property P.

Caution: The diagram

$$\begin{tikzcd} \underline{X} \arrow[r, "i_X"] \arrow[d, "\underline{f}"'] & LOG_X \arrow[d, "LOG(f)"] \\ \underline{S} \arrow[r, "i_S"'] & LOG_S \end{tikzcd}$$

does not necessarily commute. It commutes if and only if f is strict.

Recall that a morphism of fine log schemes $f : X \to S$ is said to be *formally log smooth* (resp. *formally log étale*) if in the following commutative diagram of fine log schemes, étale locally on T there exists (resp. exists a unique) a lifting

$$\begin{tikzcd} T_0 \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

for any strict square zero thickening $i : T_0 \to T$. We say that f is *log smooth* (resp. *log étale*) if it is formally log smooth (resp. formally log étale) and $\underline{f}$ is locally of finite presentation. One defines (formal) smoothness and étaleness for representable morphisms of algebraic stacks in the similar way.

Theorem 6.6. *For a morphism $f : X \to S$ of fine log schemes, f is LOG smooth (resp. LOG étale) if and only if f is log smooth (resp. log étale), if and only if f is weakly LOG smooth (resp. weakly LOG étale).*

refer back here to earlier discussion

This is part of [32, 4.6].

Another application is the following. Let $f_0 : X_0 \to T_0$ be a log smooth integral morphism of fine log schemes, and $T_0 \xrightarrow{i} T$ a strict square zero thickening defined by the ideal $I \subset \mathcal{O}_T$. Then we can consider the deformation problem of classifying Cartesian squares

$$\begin{tikzcd} X_0 \arrow[r, dotted] \arrow[d, "f_0"'] & X \arrow[d, dotted, "f"] \\ T_0 \arrow[r, "i"'] & T, \end{tikzcd}$$

why in ( )?

compare notation with what was used earlier

where $f : X \to T$ is also log smooth (integral). Then the solution to this deformation problem is the cohomology of the *log tangent bundle* $T^{\log}_{X_0/T_0}$, namely

- there is an obstruction class $o \in H^2(\underline{X_0}, T^{\log}_{X_0/T_0} \otimes f_0^* I)$;
- when the obstruction class vanishes so that deformations exist, the set of all isomorphism classes of deformations is a torsor under $H^1(\underline{X_0}, T^{\log}_{X_0/T_0} \otimes f_0^* I)$;
- the automorphism group of each deformation is $H^0(\underline{X_0}, T^{\log}_{X_0/T_0} \otimes f_0^* I)$.

The morphism $f_0 : X_0 \to T_0$ of fine log schemes gives a smooth morphism $\underline{X_0} \xrightarrow{f_0} LOG_{T_0}$, also denoted f_0, and the above deformation problem is equivalent to the following

$$\begin{tikzcd} \underline{X_0} \arrow[r, dotted] \arrow[d, "f_0"'] & \underline{X} \arrow[d, dotted, "f"] \\ LOG_{T_0} \arrow[r] & LOG_T. \end{tikzcd}$$

The solution to this deformation problem is the cohomology of the *ordinary tangent bundle* $T_{\underline{X_0}/LOG_{T_0}}$. In fact, we have $\Omega^{\log}_{X_0/T_0} \cong \Omega_{\underline{X_0}/LOG_{T_0}}$, because they represent the same functor. (insert ref for more details).

Finally, the stack LOG_S can be covered by toric stacks discussed above. Let S be a fine log scheme and $U \to S$ an étale map, and U is given the inverse image log structure. Let $Q \xrightarrow{\beta} \mathcal{M}_U$ be a chart, and $h : Q \to P$ a morphism of fine monoids, then we have a natural morphism $\mathcal{S}_P \times_{\mathcal{S}_Q} \underline{U} \to LOG_U$.

why? what is the map?

Proposition 6.7. *For any fine log scheme S, the natural morphism*

$$\coprod_{(U,\beta,h)} \mathcal{S}_P \times_{\mathcal{S}_Q} \underline{U} \to LOG_S$$

is a representable étale surjection, where the disjoint union is taken over all triples (U, β, h) consisting of an étale morphism $U \to S$, a chart $Q \xrightarrow{\beta} \mathcal{M}_U$, and a morphism $h : Q \to P$ of fine monoids, for some fine monoids P and Q.

LogDef

7. Log deformation theory in general

7.1. Introduction

As we know, for schemes, the general deformation theory is not as easy as the smooth case. To understand deformation theory of general morphisms, one has to use the full power of the cotangent complex. In log geometry, one can generalize it to get a reasonable theory of logarithmic cotangent complex.

ref to Illusie.

This log cotangent complex will be compatible with the usual cotangent complex when the morphism in question is strict and is also compatible with log smooth deformation theory for log smooth morphisms. Basically, this is an application of the deformation theory of representable morphisms to algebraic stacks ([36]) to the classifying morphisms from the underlying scheme of X to the stack LOG_Y ([32], see also section 6 of this chapter.)

Convention. We will focus on the category of fine log schemes, and the functors will be always in the derived sense. For a log scheme X, $\underline{X}$ means the underlying scheme of X.

?

$\underline X$

Remark. We will work with the category $D'(\underline{X}_{\acute{e}t})$ and similar categories, and one can talk about distinguished triangles and Ext's in these categories. For relevant definitions, see [35] and [36].

some review of D' should be given (and in fact it is not necessary anymore).

Our presentation here follows [35].

precise refs:

7.2. The Log Cotangent Complex

In [32], an Artin stack LOG_Y is defined so that to give a morphism of log schemes $X \to Y$ is equivalent to give a representable morphism $\underline{X} \to LOG_Y$. Thus one may think deformations of the morphism of log schemes $X \to Y$ as deformations of the representable morphism from $\underline{X}$ to the stack LOG_Y. In [36], the deformation theory of such morphisms was studied in detail. As an application of this theory, one makes the following definition:

refer back to §6.

Definition 7.1. For a morphism of log schemes $f : X \to Y$, the logarithemic cotangent complex of f is defined by $L_f = L_{\underline{X}/LOG_Y}$, where the right hand side is the cotangent complex of the morphism $\underline{X} \to LOG_Y$ defined in [36].

precise ref.

Remark. One should think about L_f as an object of the category $D'_{qcoh}(\underline{X}_{\acute{e}t})$. In the above definition, the right hand side is an object of the category $D'_{qcoh}(\underline{X}_{lis-\acute{e}t})$. As the restriction functor $D'_{qcoh}(\underline{X}_{lis-\acute{e}t}) \to D'_{qcoh}(\underline{X}_{\acute{e}t})$ is an equivalence of categories, no information of the cotangent complex would lost.

7.3. Basic Properties

For every morphism of fine log schemes $f : X \to Y$ the log cotangent complex is a projective system

$$L_f = (\cdots \to L_f^{\geq -n-1} \to L_f^{\geq -n} \to \cdots \to L_f^{\geq 0})$$

where each $L_f^{\geq -n}$ is an essentially constant ind-object in $D^{[-n,0]}(\mathcal{O}_X)$ (The derived category of $\mathcal{O}_X$-modules supported in $[-n, 0]$).

The log cotangent complex L_f has the following properties:

(1) For any $n \geq 0$, the natural map $\tau_{\geq -n} L_f^{\geq -n-1} \to L_f^{\geq -n}$ is an isomorphism.
(2) If f is strict, then the system $(\tau_{\geq -n} L_{f'})$ represents L_f, where $L_{f'}$ is the usual cotangent complex of the underlying morphism of schemes f'.
(3) If $f : X \to Y$ is log smooth, then the sheaf of log differentials $\Omega^1_{X/Y}$ represents L_f. logsm
(4) If

$$\begin{tikzcd} X' \arrow[r, "a"] \arrow[d, "g"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "b"] & Y \end{tikzcd}$$

is a commutative diagram of fine log schemes, then there is a natural map

$$a^* L_f \to L_g$$

which is an isomorphism if the square above is cartesian and f is log flat. Furthermore, if the composite $X' \to Y' \to Y$ satisfies the condition (T) below, then the map

$$g^*L_b \oplus a^*L_f \to L_{bg}$$

is also an isomorphism.

(5) Given a composite

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

fix numbering

satisfying condition (T) below, there is a natural map

$$L_f \to f^*L_g[1]$$

making the resulting triangle

tri (7.2)
$$f^*L_g \to L_{gf} \to L_f \to f^*L_g[1]$$

distinguished.

Remark. One might hope a theory of log cotangent complex in which every triangle (7.2) is distinguished. However, such a theory is too good to exist. where?

On the other hand, Gabber has shown that if one loosen the requirement 3, then one can obtain a theory of log cotangent complexes for which one has a distinguished triangle (7.2) for all composite $X \xrightarrow{f} Y \xrightarrow{g} Z$.

The Condition (T) mentioned above is the following:
There exists a family of commutative diagrams

$$\begin{tikzcd}
X_i \arrow[r, "\pi_{X_i}"] \arrow[dr] & X \times_Y Y_i \arrow[r] \arrow[d] & X \times_Z Z_i \arrow[r] \arrow[d] & X \arrow[d, "f"] \\
& Y_i \arrow[r, "\pi_{Y_i}"] \arrow[dr] & Y \times_Z Z_i \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\
& & Z_i \arrow[r, "\pi_{Z_i}"] & Z
\end{tikzcd}$$

such that

(1) The underlying schemes of X_i, Y_i, Z_i are all affine.
(2) The π's are all strict, and their underlying morphisms are flat and locally of finite presentation.
(3) The underlying family of morphisms of schemes of $\{X_i \to X\}$ ~~are~~ is jointly surjective.
(4) There exists charts

slightly sloppy

$$\beta_{X_i} : Q_{X_i} \to \mathcal{M}_{X_i}, \beta_{Y_i} : Q_{Y_i} \to \mathcal{M}_{Y_i}, \beta_{Z_i} : Q_{Z_i} \to \mathcal{M}_{Z_i}$$

and injective maps

$$Q_{Z_i} \to Q_{Y_i} \to Q_{X_i}$$

compatible with the morphisms f_i, g_i and

$$\mathrm{Tor}^j_{\mathcal{O}_{Z_i}\otimes_{\mathbb{Z}[Q_{Z_i}]}\mathbb{Z}[Q_{Y_i}]}(\mathcal{O}_{Z_i}\otimes_{\mathbb{Z}[Q_{Z_i}]}\mathbb{Z}[Q_{X_i}], \mathcal{O}_{Y_i}[G]) = 0 \text{ for all } j > 0.$$

Here $G := \mathrm{Coker}(Q^{gp}_{Z_i} \to Q^{gp}_{Y_i})$ and $\mathcal{O}_{Y_i}[G]$ is viewed as an $\mathcal{O}_{Z_i}\otimes_{\mathbb{Z}[Q_{Z_i}]}\mathbb{Z}[Q_{Y_i}]$-algebra via the map

some examples?

$$\mathcal{O}_{Z_i}\otimes_{\mathbb{Z}[Q_{Z_i}]}\mathbb{Z}[Q_{Y_i}] \to \mathcal{O}_{Y_i}[G], t\otimes e_q \mapsto g_i^*(t)\beta_{Q_{Y_i}}(q)\cdot\bar{q}$$

where $\bar{q}$ denotes the image of q in G.

7.4. Deformation Theory of Log Schemes in General

In this section, we explain the relation between the log cotangent complex and deformation theory of log schemes. Let $f : X \to Y$ be a morphism of fine log schemes and let I be a quasi-coherent sheaf on X. Define a Y-extension of X by I to be a commutative diagram of log schemes

strict?

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "f"'] & X' \arrow[dl, "f'"] \\ Y & \end{tikzcd}$$

where j is an exact closed immersion (meaning that the log structure on X is isomorphic to the pull-back of that of X''s) defined by a square-zero ideal, together with an isomorphism $\epsilon_j : I \cong \mathrm{Ker}(\mathcal{O}_{X'} \to \mathcal{O}_X)$. The set of Y extensions by I forms in a natural way a category $\underline{\mathrm{Exal}}_Y(X, I)$. Let $\mathrm{Exal}_Y(X, I)$ be the set of isomorphism classes of this category.

There is a tautological equivalence of categories (see [36] for the meaning of the right hand side):

review briefly def.

$$\underline{\mathrm{Exal}}_Y(X, I) \cong \underline{\mathrm{Exal}}_{LOG_Y}(\underline{X}, I).$$

Hence by ([36], 1.1) and our definition of L_f we obtain the following result:

Theorem 7.3. *There is a natural bijection*

$$\mathrm{Exal}_Y(X, I) \cong \mathrm{Ext}^1(L_f, I).$$

It is precisely the theorem above that guarantees that general deformation theory is controlled by our logarithmic cotangent complex.

8.1. What is rounding?

The process of "rounding," in its most basic form, produces a manifold *with corners* from a *smooth* analytic space with a normal crossings divisor. Evidently then, "rounding" is something of a misnomer. In various moduli problems, the rounding of the moduli space often has a more natural topological interpretation than the moduli space itself. For example, the relative Hilbert stack of a marked Riemann surface can be defined algebraically using Jun Li style expansions, but it is not representable (except in some trivial cases), but its rounding is a topological space (a manifold with corners, even) which is relatively easy to describe. A similar phenomenon occurs in many moduli problems involving expansions. Another example is the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of marked nodal curves, whose boundary is a normal crossings divisor. The "interior" $\mathcal{M}_{g,n}$ can be described topologically as a quotient of Teichmuller space by the appropriate mapping class group. There are generalizations of Teichmuller space and the mapping class group involving 2-manifolds decorated with circles; the analogous quotient yields a topological description of the *rounding* of $\overline{\mathcal{M}}_{g,n}$, rather than the moduli space $\overline{\mathcal{M}}_{g,n}$ itself.

references?

The topological preeminence of the rounding of moduli spaces might ultimately be traced back to the preference in topology for operations involving real codimension one subspaces (e.g. connected sum of manifolds) as opposed to the algebro-geometric preference for complex codimension one operations (e.g. pushout of two smooth varieties along a common divisor). From this point of view, one might think of, say, log geometry as an attempt to speak algebraically about various "real codimension one" phenomena.

8.2. The oriented real blowup

The most general rounding operation is the *Kato–Nakayama space* associated to a log analytic space. In the basic example of a smooth analytic space with log structure from a normal crossings divisor, the Kato–Nakayama space can be described in terms of *oriented real blowup*, which is a relatively simple rounding operation that can be described as follows. Suppose X is a topological space, $\pi : L \to X$ is a complex line bundle, and $s : X \to L$ is a section of π. Locally on X we can choose a trivialization $(\pi, \phi) : L \to X \times \mathbb{C}$ and consider the subspace

$$\mathrm{B}_{L,s,\phi} X := \left\{ l \in L \ : \ |\phi(l)| \cdot (\phi s \pi)(l) \ = \ \phi(l) \cdot |(\phi s \pi)(l)| \right\}$$

of L. A continuous function $u : X \to \mathbb{C}^*$ yields a new trivialization $(\pi, u \cdot \phi)$, where $(u \cdot \phi)(l) := (u\pi)(l)\phi(l)$. The key observation is that $\mathrm{B}_{L,s,\phi} X = \mathrm{B}_{L,s,u\cdot\phi} X$, so the subspace $\mathrm{B}_{L,s,\phi} X$ is independent of the choice of ϕ, hence one can define a subspace $\mathrm{B}_{L,s} X \subseteq L$ by defining it locally on X using a trivialization, then gluing the locally defined subspaces. From the local picture using a trivialization, it is clear that the subspace $\mathrm{B}_{L,s} X$ contains the zero section and $L|_{Z(s)}$ (where $Z(s) \subseteq X$ is the zero locus of s) and is invariant under the $\mathbb{R}_{>0}$ action on L inherited from the full $\mathbb{C}^*$ scaling action. We let $\mathrm{B}^*_{L,s} X$ be the complement of the zero section in $\mathrm{B}_{L,s}$ and we call

$$\mathrm{Blo}_{L,s} X := (\mathrm{B}^*_{L,s} X)/\mathbb{R}_{>0}$$

the *oriented real blowup* of X along (L, s).

The space $\mathrm{Blo}_{L,s} X$ is a closed subspace of the oriented circle bundle $S^1 L := L^*/\mathbb{R}_{>0}$ associated to L and is, in particular, proper over X. The projection $\tau : \mathrm{Blo}_{L,s} X \to X$ is an isomorphism away from $Z(s)$ and $\tau^{-1}(Z(s))$ is oriented circle bundle $S^1 L|_{Z(s)}$. The spaces $\mathrm{B}_{L,s} X$ and $\mathrm{Blo}_{L,s} X$ are natural under pulling back line bundles and sections.

If X is an analytic space and $D \subseteq X$ is a Cartier divisor, then D determines a line bundle $\mathcal{O}_X(D)$ together with a section s whose zero locus is D. In this situation, we will write $\mathrm{B}_D X$, $\mathrm{Blo}_D X$, etc. and speak of the *oriented real blowup of X along D*. The space $\mathrm{Blo}_D X$ inherits a differentiable structure from its inclusion in $S^1 \mathcal{O}_X(D)$.

The basic example to keep in mind is the oriented real blowup $\mathrm{Blo}_0 \mathbb{C}$ of the complex plane $\mathbb{C}$ at the origin. The origin is the zero locus of the identity map $\mathrm{Id} : \mathbb{C} \to \mathbb{C}$, hence

$$\begin{aligned}\mathrm{Blo}_0 \mathbb{C} &= \{(z, Z) \in \mathbb{C} \times \mathbb{C}^* : |z| Z = z|Z|\} \;/\; \mathbb{R}_{>0} \\ &= \{(z, Z) \in \mathbb{C} \times S^1 : |z| Z = z\} \\ &\cong \mathbb{R}_{\geq 0} \times S^1,\end{aligned}$$

where the last isomorphism from $\mathbb{R}_{\geq 0} \times S^1$ is given by $(\lambda, Z) \mapsto (\lambda Z, Z)$. Evidently $\mathrm{Blo}_0 \mathbb{C}$ is a half-infinite annulus whose boundary S^1 is the exceptional locus of $\tau : \mathrm{Blo}_0 \mathbb{C} \to \mathbb{C}$ (the fiber over the origin).

8.3. The Kato–Nakayama space

Suppose X is an analytic space and $D \subseteq X$ is a Cartier divisor. Let

$$\mathcal{M}_X := \left\{ f \in \mathcal{O}_X \;:\; f|_{X \setminus D} \in \mathcal{O}^*_{X \setminus D} \right\}$$

better to use $X^{\log}$ — ref to KNN

be the associated log structure. The *Kato–Nakayama space* X^{KN} associated to the log analytic space $(X, \mathcal{M}_X)$ has as points pairs (x, F) where $x \in X$ and $F : \mathcal{M}_{X,x} \to S^1$ is a monoid homomorphism satisfying $F(u) = u(x)/|u(x)|$ for every $u \in \mathcal{O}^*_{X,x} \subseteq \mathcal{M}_{X,x}$. This set is topologized as follows: Locally on X we can find $f_1, \ldots, f_n \in \mathcal{M}_X(X)$ which, together with the units, generate $\mathcal{M}_X$. The map

$$\begin{aligned} X^{\mathrm{KN}} &\to X \times (S^1)^n \\ (x, F) &\mapsto (x, F(f_{1,x}), \ldots, F(f_{n,x})) \end{aligned}$$

is then easily seen to be a monomorphism onto a closed subset of $X \times (S^1)^n$, so we give X^{KN} the subspace topology so that this is a closed embedding. Since one can check easily that this topology doesn't depend on the choice of generators $f_1, \ldots, f_n$, the locally defined topologies glue to a topology on X^{KN} making the projection $\tau : X^{\mathrm{KN}} \to X$ given by $\tau(x, F) := x$ a proper map.

In general, one can define a Kato–Nakayama space X^{KN} for any (fine) log scheme in roughly the same manner. The Kato–Nakayama space is a model for the log de Rham cohomology of X in the sense that

$$\mathrm{H}^*(X^{\mathrm{KN}}, \mathbb{C}) = \mathbb{H}^*(X, \wedge^\bullet \Omega_X^\dagger)$$

under mild assumptions on X.

8.4. Relating Kato–Nakayama spaces to oriented blowups

There is a morphism

$$\begin{aligned} \phi : X^{\mathrm{KN}} &\to \mathrm{Blo}_D X \\ (x, F) &\mapsto (x, f \mapsto F(\overline{f})) \end{aligned}$$

of topological spaces over X which requires a little explanation. Here $f \in S^1\mathcal{O}_X(-D)|_x$ is in the circle bundle associated to the fiber $\mathcal{O}_X(-D)|_x \cong \mathbb{C}$ and $\overline{f} \in \mathcal{O}_{X,x}(-D)$ is a lifting of f to the stalk (one shows that any such $\overline{f}$ is actually in $\mathcal{M}_{X,x}$ and that $F(\overline{f})$ doesn't depend on this choice of lifting $\overline{f}$). If we use the identification

$$S^1\mathcal{O}_X(D) = \mathrm{Hom}_{S^1}(S^1\mathcal{O}_X(-D), S^1),$$

then we can think of ϕ as a map from X^{KN} to $S^1\mathcal{O}_X(D)$; one then shows that this ϕ is continuous and that ϕ factors through $\mathrm{Blo}_D X \subseteq S^1\mathcal{O}_X(D)$.

When X is a smooth analytic space and D is a smooth divisor, the map ϕ is easily seen to be an isomorphism since one can reduce to the case $(X, D) = (\mathbb{C}, 0)$ on formal grounds. Slightly more generally, if X is smooth, but D is only a normal crossings divisor, then locally we can write D as a union of smooth divisors $D_1, \ldots, D_i$ which look like the first i coordinate hyperplanes in $\mathbb{C}^n$ ($n = \dim X$), and we can define a variant of the oriented real blowup

$$\mathrm{Blo}'_D X := (\mathrm{Blo}_{D_1} X) \times_X \cdots \times_X (\mathrm{Blo}_{D_i} X)$$

and a map $\phi : X^{\mathrm{KN}} \to \mathrm{Blo}'_D X$. In this local picture, the log structure $\mathcal{M}_X$ is the direct sum (in the category of log structures) of the log structures $\mathcal{M}_X^j$ from $D_1, \ldots, D_i$, the associated Kato–Nakayama space X^{KN} is the fibered product over X of the $(X, \mathcal{M}_X^j)^{\mathrm{KN}}$, and ϕ is just the fibered product over X of the previously constructed isomorphisms $\phi_j : (X, \mathcal{M}_X^j)^{\mathrm{KN}} \to \mathrm{Blo}_{D_j} X$. The locally defined variants can be glued to define a global variant $\mathrm{Blo}'_D X$ of the oriented real blowup and an isomorphism $\phi : X^{\mathrm{KN}} \cong \mathrm{Blo}'_D X$ of topological spaces over X.

8.5. Topology and the Kato–Nakayama space

Locally, if $X = \mathbb{C}^n$ and D is the union of the first i coordinate hyperplanes, then D is the zero locus of $(z_1, \dots, z_n) \mapsto z_1 \cdots z_i \in \mathbb{C}$ and we have

$$\begin{aligned} \operatorname{Blo}_D X &= \{(z_1, \dots, z_n, Z) \in \mathbb{C}^n \times S^1 : |z_1 \cdots z_i| Z = z_1 \cdots z_i\} \\ \operatorname{Blo}'_D X &= \{(\overline{z}, \overline{Z}) \in \mathbb{C}^n \times (S^1)^i : |z_j| Z_j = z_j \text{ for } j = 1, \dots, i\}. \end{aligned}$$

In the general normal crossings divisor situation, the fiber of $\tau : X^{\mathrm{KN}} \to X$ over a point $x \in X$ is naturally identified with

$$S^1 N_{D_1/X}|_x \times \cdots \times S^1 N_{D_i/X}|_x,$$

where $D_1, \dots, D_i$ are the branches of D containing x. When $\dim X = n$, a point $y \in \tau^{-1}(x)$ has a neighborhood diffeomorphic to a neighborhood of the origin in $\mathbb{R}^i_{\geq 0} \times \mathbb{R}^{2n-i}$. (Note $i \leq n$, so the depth of the corners in a Kato–Nakayama space is somewhat constrained.) Recall that the *topology* near the origin only depends on whether $i > 0$, but the differentiable structure depends on the actual value of i. In particular, the topological boundary of the manifold X^{KN} is given by $\tau^{-1}(D)$, and this manifold with boundary is homotopy equivalent to its interior, so $\mathrm{H}^*(X^{\mathrm{KN}}) = \mathrm{H}^*(X \setminus D)$.

8.6. Kato–Nakayama spaces of expanded pairs

Given a pair (X, D) consisting of a smooth variety X over $\mathbb{C}$ with a smooth divisor $D \subseteq X$, the notion of an *expanded pair* $t : \mathcal{X} \to B$ over a base B arises in various relative curve counting theories. The fiber of t over a point $b \in B$ always looks like

$$X[n]_0 = X \coprod_D \Delta_1 \coprod_D \cdots \coprod_D \Delta_n$$

(for an appropriate n), where $\Delta_i = \mathbb{P}(N_{D/X} \oplus \mathcal{O}_D)$ is a $\mathbb{P}^1$ bundle over X. Both $\mathcal{X}$ and B have natural log structures making t a log smooth map of log schemes. The fiber of $t^{\mathrm{KN}} : \mathcal{X}^{\mathrm{KN}} \to B^{\mathrm{KN}}$ over a point $c \in \tau_B^{-1}(b)$ looks like

$$X^{\mathrm{KN}} \coprod_{c_1 : S^1 N_{D/X} \cong S^1 N_{D/\Delta_1}} \Delta_1^{\mathrm{KN}} \cdots \coprod_{c_n : S^1 N_{D/\Delta_{n-1}} \cong S^1 N_{D/\Delta_n}} \Delta_n^{\mathrm{KN}},$$

where the choice of $c \in \tau^{-1}(b) \cong (S^1)^n$ determines the choice of orientation reversing S^1 bundle isomorphisms $c_1, \dots, c_n$. Here each Δ_j has the log structure from the two copies of D, and Δ_j^{KN} is a cylinder bundle over D (better: an I-bundle over D^{KN}).

The action of $(\mathbb{C}^*)^n$ on $X[n]_0$ given by scaling the fibers of the $\mathbb{P}^1$ bundles Δ_i is an action by isomorphisms of log schemes, so it lifts to an action on Kato–Nakayama spaces. This lifted action is nontrivial on B^{KN} as the $(S^1)^n$ factor of $(\mathbb{C}^*)^n$ acts simply transitively on $\tau_B^{-1}(b) \cong (S^1)^n$. In the usual moduli problems involving expansions, the isotropy group of a point b involves elements of $(\mathbb{C}^*)^n$

such that the induced action on $X[n]_0$ respects a map from a curve to $X[n]_0$, a subscheme of $X[n]_0$, etc., and this isotropy is usually required to be finite to have a good moduli problem. Since $G \cap \mathbb{R}_{>0} = \{\mathrm{Id}\}$ for any finite subgroup G of $\mathbb{C}^*$, the Kato–Nakayama space of the moduli problem is often representable even if the moduli problem itself is not. This is always the case for moduli problems involving, say, quotients of sheaves on $X[n]_0$ pulled back from X, since these quotients themselves have no automorphisms and the only isotropy comes from the subgroup of $(\mathbb{C}^*)^n$ preserving the quotient.

Should this section be called
- 'Kato-Nakayama spaces'
- mention relation w/ Betti/de Rham coh
- work of Ogus-Nakayama?

Definition 7.4. Let $j_0 : Y_0 \hookrightarrow Y$ be an ~~exact~~ closed immersion of fine log schemes defined by a square-zero ideal $I \subset \mathcal{O}_Y$, and let $f_0 : X_0 \to Y_0$ be a log flat morphism. A log flat deformation of X_0 to Y is a cartesian square

strict

where has this been defined (I suppose in §6)

$$\begin{tikzcd} X_0 \arrow[r, "j"] \arrow[d, "f_0"'] & X \arrow[d, "f"] \\ Y_0 \arrow[r, "j_0"] & Y \end{tikzcd}$$

with f log flat.

To give a log flat deformation as above is equivalent to give a 2-commutative diagram

$$\begin{tikzcd} \underline{X_0} \arrow[r, "j"] \arrow[d, "\mathcal{L}_{f_0}"'] & \underline{X} \arrow[d, "\mathcal{L}_f"] \\ LOG_{Y_0} \arrow[r, "j_0"] & LOG_Y \end{tikzcd}$$

with $\mathcal{L}_f$ flat. Thus from ([36], 1.4) we obtain the following:

Theorem 7.5. *Let J denote the ideal of LOG_{Y_0} in LOG_Y. Then*

(1) *There exists a canonical class $o \in \mathrm{Ext}^2(L_{f_0}, \mathcal{L}^*_{f_0} J)$ whose vanishing is equivalent to the existence of a log flat deformation of X_0 to Y.*
(2) *If $o = 0$, then the set of isomorphism classes of log flat deformations of X_0 to Y is naturally a torsor under $\mathrm{Ext}^1(L_{f_0}, \mathcal{L}^*_{f_0} J)$.*
(3) *The automorphism group of any log flat deformation of X_0 to Y is canonically isomorphic to $\mathrm{Ext}^0(L_{f_0}, \mathcal{L}^*_{f_0} J))$.*

This theorem gives a satisfactory answer to the question of general deformation theory of log schemes.

Deformations of morphisms?

Rounding

8. Rounding

DeRham

9. Log De Rham and hodge structures

9.1. Moduli spaces of polarized Hodge structures.

This section owes much to a lecture by Phillip Griffiths [10].

First of all, we briefly summarize the classical theory of the moduli spaces of polarized Hodge structures.

subsection1.1

9.1.1. The moduli space $M_h = \Gamma \backslash D_h$. Let n be an integer, and let h be a sequence of positive integers $(h^{n,0}, h^{n-1,0}, \cdots, h^{0,n})$ satisfying $h^{p,q} = h^{q,p}$, called the *Hodge numbers*. Let $H_{\mathbb{Z}}$ be a free abelian group of rank $\sum h^{p,n-p}$, with a non-degenerate bilinear form $Q : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \to \mathbb{Z}$, which is symmetric (resp. anti-symmetric) if n is even (resp. odd). Let $G_{\mathbb{Z}}$ be the group functor $\mathrm{Aut}(H_{\mathbb{Z}}, Q)$ on rings, sending a ring R to the group of automorphisms on the free R-module either review these notions or give precise ref.

$H_R := H_{\mathbb{Z}} \otimes R$ preserving the bilinear form Q. It is clearly a group scheme over $\mathbb{Z}$. Let Γ be an arithmetic subgroup of $G_{\mathbb{Z}}(\mathbb{Z})$ ([28], §3).

?

The set of Hodge structures of weight n on $H_{\mathbb{R}}$ with prescribed Hodge numbers h, such that Q induces a *polarization* on $H_{\mathbb{R}}$ (i.e. it induces a morphism $H_{\mathbb{R}} \otimes H_{\mathbb{R}} \to \mathbb{R}(-n)$ of Hodge structures, and the bilinear form $Q_C(u,v) := Q(u, Cv)$, where C is the Weil operator, is symmetric and positive definite), is parameterized by the homogeneous space $D_h = G_{\mathbb{R}}/K$, where K is the stabilizer group of a fixed polarized Hodge structure F_0 on $H_{\mathbb{R}}$.

briefly say what this means?

This homogeneous space $D = D_h = G_{\mathbb{R}}/K$ has a complex structure defined as follows. It is clear that $Q : H_{\mathbb{R}} \otimes H_{\mathbb{R}} \to \mathbb{R}(-n)$ is a morphism of Hodge structures if and only if $Q(F^p, F^{n-p+1}) = 0$ for all p. Let $f^p = \sum_{r \geq p} h^{r,n-r}$, and let $D^\vee$, the *compact dual of* D, to be the subspace of the product of the Grassmannians $\prod_p \mathrm{Gr}(f^p, H_{\mathbb{R}})$ consisting of all flags $F^\bullet$

$$\cdots \subset F^{p+1} \subset F^p \subset \cdots$$

such that $Q(F^p, F^{n-p+1}) = 0$. Then $D^\vee = G_{\mathbb{C}}/P$, where P is a parabolic subgroup corresponding to a fixed flag. This gives $D^\vee$ a complex structure. We see that $D \subset D^\vee$ is the locus of flags satisfying

(i) $F^p \cap \overline{F}^{n-p+1} = 0$ (so that $F^p \oplus \overline{F}^{n-p+1} \cong H_{\mathbb{C}}$), and

(ii) $Q(\overline{u}, Cu) > 0$ for $u \neq 0$ in $H_{\mathbb{C}}$.

They are both open conditions, so $D \subset D^\vee$ is an open complex submanifold. The group Γ acts on D_h properly discontinuously, and the quotient $M_h = \Gamma \backslash D_h$ is the moduli space of *Γ-equivalence classes of Q-polarized Hodge structures on $H_{\mathbb{C}}$ with Hodge type h*. See ([20], 0.3.6, 0.3.7).

examples? A_g, M_g...

9.1.2. Variations of Hodge structure.

Definition 9.1. Let S be a complex manifold. A *variation of Hodge structure $\mathscr{H}$ of weight n on S* is given by

- a local system $\mathscr{H}_{\mathbb{Z}}$ of free abelian groups of finite rank on S;
- a finite decreasing filtration $F^\bullet \mathscr{H}_{\mathscr{O}}$ of the vector bundle $\mathscr{H}_{\mathscr{O}} := \mathscr{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathscr{O}_S$ by holomorphic sub-bundles,

fix numbering.

such that the following conditions are satisfied:

1) (Griffiths transversality) the natural flat connection $\nabla = d \otimes id_{\mathscr{H}_{\mathbb{Z}}} : \mathscr{H}_{\mathscr{O}} \to \Omega^1_S \otimes \mathscr{H}_{\mathscr{O}}$ takes $F^p \mathscr{H}_{\mathscr{O}}$ into $\Omega^1_S \otimes F^{p-1}\mathscr{H}_{\mathscr{O}}$, for every p;

2) for each point $s \in S$, the fiber $F^\bullet(s)$ over s is a Hodge structure of weight n.

A *polarization* of the variation of Hodge structure $\mathscr{H}$ is a locally constant bilinear form

isn't loc. constant automatic?

$$\mathscr{Q} : \mathscr{H}_{\mathbb{Z}} \otimes \mathscr{H}_{\mathbb{Z}} \to \underline{\mathbb{Z}}$$

such that on each fiber over $s \in S$, it induces a polarization of the fiber Hodge structure.

Suppose we have a polarized family of Hodge structures $(\mathscr{H}, \mathscr{Q} : \mathscr{H}_{\mathbb{Z}} \otimes \mathscr{H}_{\mathbb{Z}} \to \mathbb{Z})$ of weight n on S, and a global section of the sheaf

$$\Gamma \backslash \underline{\mathrm{Isom}}((\mathscr{H}_{\mathbb{Z}}, \mathscr{Q}), (H_{\mathbb{Z}}, Q)),$$

where $H_{\mathbb{Z}}$ is regarded as a constant sheaf on S, and assume that all the monodromies of this family of Hodge structures on S are contained in Γ. Then there is a *period map*

what does this mean?

$$\varphi : S \to M_h$$

inducing this family of Hodge structures. This map is locally liftable to D_h.

how defined defn?

Too many things used without def'g refs.

If $f : X \to S$ is a projective smooth morphism between quasi-projective complex algebraic manifolds, with a relative hyperplane section $\eta \in H^0(S, R^2 f_* \mathbb{Z})$, then the family of the primitive part $P^n(X_s, \mathbb{Z})$ of the cohomology groups $H^n(X_s, \mathbb{Z})$ modulo torsion form a polarized variation of Hodge structure of weight n on S, and it induces a period map $S \to M_h$. To be precise, the family of $H^n(X_s, \mathbb{C})$'s are the stalks of $R^n f_*(f^{-1}\mathscr{O}_S)$, and the Hodge filtration on $R^n f_*(f^{-1}\mathscr{O}_S)$ is given by the degenerate spectral sequence

$$E_1^{pq} = R^q f_* \Omega^p_{X/S} \Longrightarrow R^{p+q} f_*(f^{-1}\mathscr{O}_S),$$

which is induced from the resolution $\Omega^\bullet_{X/S}$ of $f^{-1}\mathscr{O}_S$ (the relative holomorphic Poincaré lemma, see ([7], 3.4)). Since η is a global section, the primitive part form a variation of sub-Hodge structures on S.

9.2. Logarithmic Hodge structures.

One can ask the following question. Let $f : X \to S$ be a family of projective manifolds, and let S be the complement of a normal crossing divisor D in some compact manifold $\overline{S}$, and suppose one can extend the family f to a family $\overline{f} : \overline{X} \to \overline{S}$ which is log smooth (here $\overline{S}$ has the log structure induced by the divisor D). Is it possible to enlarge the moduli space M_h to some $\overline{M}_h$ so that the period map extends to $\overline{\varphi} : \overline{S} \to \overline{M}_h$?

To study the degenerations of Hodge structures, Kato and Usui introduced the notion of logarithmic Hodge structures. (ref).

9.2.1. The ringed space $X^{\log}$. Let $(X, \alpha : M_X \to \mathscr{O}_X)$ be an fs log analytic space over $\mathbb{C}$ (for instance the $\mathbb{C}$-points of an fs log scheme over $\mathbb{C}$), and let $X^{\log}$ be the set of pairs (x, u), where $x \in X$ and $u : M_{X,x} \to S^1$ is a homomorphism of monoids, such that $u(f) = f(x)/|f(x)|$ for $f \in \mathscr{O}^*_{X,x} \subset M_{X,x}$. Here S^1 is the unit circle in the complex plane. Let $\tau : X^{\log} \to X$ be the function $(x, u) \mapsto x$. For any open $U \subset X$ and $f \in M_X(U)$, there is a function $arg(f) : \tau^{-1}(U) \to S^1$ sending $(x, u) \mapsto u(f)$. We give $X^{\log}$ the weakest topology such that the functions τ and $arg(f)$ are continuous. Over the open set $X^* \subset X$ where the log structure is trivial, the map τ is a homeomorphism, and the section $j^{\log} : X^* \hookrightarrow X^{\log}$ is examples?

a homotopy equivalence. The map τ is proper, with fibers $\tau^{-1}(x)$ compact tori $(S^1)^m$, where m is the rank of $\overline{M}^{\mathrm{gp}}_{X,x}$.

~~what properties are needed of $\mathcal{O}_X$ log & why Steps to define~~

One can define a sheaf of rings $\mathscr{O}_{X^{\log}}$ on $X^{\log}$. Roughly speaking, this is the subsheaf of rings of $j_*^{\log}\mathscr{O}_{X^*}$ on $X^{\log}$ generated over $\tau^{-1}\mathscr{O}_X$ by "$\log(q)$", for all $q \in M_X^{\mathrm{gp}}$. See ([20], 2.2.4) for the precise definition.

For example, if $x \in X$ and $y \in \tau^{-1}(x)$, and the free abelian group $\overline{M}^{\mathrm{gp}}_{X,x}$ has rank m and is generated by $f_1, \cdots, f_m \in M^{\mathrm{gp}}_{X,x}$, then the stalk $\mathscr{O}_{X^{\log},y}$ is isomorphic to the polynomial ring $\mathscr{O}_{X,x}[\log(f_1), \cdots, \log(f_m)]$. This shows that in general, $(X^{\log}, \mathscr{O}_{X^{\log}})$ is not a locally ringed space.

Let Ω^1_X be the sheaf of log differential forms on the fs log analytic space X, i.e.

$$\Omega^1_X = (\Omega^1_{\underline{X}} \oplus (\mathscr{O}_X \otimes_{\mathbb{Z}} M_X^{\mathrm{gp}}))/\{(-d\alpha(f), \alpha(f) \otimes f) | f \in M_X\}.$$

refer back.

For a morphism $f : X \to Y$ of fs log analytic spaces, define

$$\Omega^1_{X/Y} = \mathrm{Coker}(f^*\Omega^1_Y \to \Omega^1_X).$$

No, should be defined directly as in alg. geom.

They are both coherent $\mathscr{O}_X$-modules. Let $\Omega^r_{X/Y}$ be the r-th exterior power of $\Omega^1_{X/Y}$, and let

$$\Omega^r_{X^{\log}/Y^{\log}} = \tau^*\Omega^r_{X/Y} = \tau^{-1}\Omega^r_{X/Y} \otimes_{\tau^{-1}\mathscr{O}_X} \mathscr{O}_{X^{\log}}.$$

One can define differential maps and have the log de Rham complex $(\Omega^\bullet_{X/Y}, d)$ (resp. $(\Omega^\bullet_{X^{\log}/Y^{\log}}, d)$) on X (resp. $X^{\log}$).

For $y \in X^{\log}$ and $x = \tau(y) \in X$, let $\mathrm{sp}(y)$ be the set of all ring homomorphisms $s : \mathscr{O}_{X^{\log},y} \to \mathbb{C}$ that extend the evaluation map $\mathrm{ev}_x : \mathscr{O}_{X,x} \to \mathbb{C}$. Since $\mathscr{O}_{X^{\log},y}$ is isomorphic to the polynomial ring over $\mathscr{O}_{X,x}$ generated by log of a basis for $\overline{M}_{X,x}$, if we fix an $s_0 \in \mathrm{sp}(y)$, then we have a bijection:

$$s \mapsto (f \mapsto s(\log(f)) - s_0(\log(f))) : \mathrm{sp}(y) \xrightarrow{\sim} Hom_{\mathrm{group}}(\overline{M}^{\mathrm{gp}}_{X,x}, \mathbb{C}).$$

9.2.2. Log variations of polarized Hodge structure.

LVPHS

Definition 9.2. Let X be an fs log analytic space. A *log variation of polarized Hodge structure of weight n on X* is given by

- a local system of free abelian groups of finite rank $\mathscr{H}_{\mathbb{Z}}$ on $X^{\log}$,
- a bilinear form $\mathscr{Q} : \mathscr{H}_{\mathbb{Z}} \otimes \mathscr{H}_{\mathbb{Z}} \to \underline{\mathbb{Z}}$,
- a finite decreasing filtration $F^\bullet\mathscr{H}_{\mathscr{O}}$ of $\mathscr{H}_{\mathscr{O}} := \mathscr{H}_{\mathbb{Z}} \otimes \mathscr{O}_{X^{\log}}$ by $\mathscr{O}_{X^{\log}}$-submodules,

such that the following conditions are satisfied:

1) there exist a locally free $\mathscr{O}_X$-module $\mathscr{E}$ and a finite decreasing filtration $F^\bullet\mathscr{E}$ by $\mathscr{O}_X$-submodules, such that $\mathrm{Gr}_p(\mathscr{E})$ is locally free for each p, and

$$F^p\mathscr{H}_{\mathscr{O}} = \tau^*F^p\mathscr{E} = \tau^{-1}F^p\mathscr{E} \otimes_{\tau^{-1}\mathscr{O}_X} \mathscr{O}_{X^{\log}};$$

2) for $y \in X^{\log}$ and $x = \tau(y) \in X$, let $s \in sp(y)$ and let $f_1, \cdots, f_r \in M_{X,x} - \mathcal{O}^*_{X,x}$ generate the monoid $\overline{M}_{X,x}$. If the $|\exp(s(\log(f_i)))|$ are sufficient small for all i, then $(\mathcal{H}_{\mathbb{Z},y}, \mathcal{Q}, F^\bullet(s))$ is a polarized Hodge structure of weight n; real numbers

3) the connection $d \otimes id : \mathcal{H}_{\mathcal{O}} \to \Omega^1_{X^{\log}} \otimes_{\mathcal{O}_{X^{\log}}} \mathcal{H}_{\mathcal{O}}$ takes $F^p\mathcal{H}_{\mathcal{O}}$ into $\Omega^1_{X^{\log}} \otimes F^{p-1}\mathcal{H}_{\mathcal{O}}$.

Here $F^\bullet(s)$, the *specialization of F at s*, is the decreasing filtration of $\mathcal{H}_{\mathbb{C},y} := \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z},y}$ defined by $F^p(s) = \mathbb{C} \otimes_{s,\mathcal{O}_{X^{\log},y}} F^p\mathcal{H}_{\mathcal{O}}$. For a fixed point $y \in X^{\log}$, the family $(\mathcal{H}_{\mathbb{Z},y}, \mathcal{Q}, F^\bullet(s))_{s \in \mathrm{sp}(y)}$ is called a *polarized log Hodge structure* on the log point $(x, M_{X,x})$; this is the same as a log variation of polarized Hodge structure on the log point $(x, M_{X,x})$.

Log variations of polarized Hodge structure arise from geometry in the following way. Let $f : X \to Y$ be a projective log smooth morphism between fs log analytic spaces, and we fix a line bundle on X which is relatively ample over Y. By a theorem of Kajiwara and Nakayama, for every integer n, the sheaf $R^n f^{\log}_* \mathbb{Z}$ is a local system on $Y^{\log}$. We take $\mathcal{H}_{\mathbb{Z}}$ to be $R^n f^{\log}_* \mathbb{Z}$ modulo torsion, take $\mathcal{Q}$ to be the pairing induced by the fixed ample line bundle, take $\mathcal{E}$ to be $R^n f_*(\Omega^\bullet_{X/Y})$, with filtration $F^p\mathcal{E} = R^n f_*(\Omega^{\geq p}_{X/Y}) \subset \mathcal{E}$, and take $F^p\mathcal{H}_{\mathcal{O}}$ to be $\tau^* F^p\mathcal{E}$. Then by a theorem of Kato, Matsubara and Nakayama, this is a log variation of polarized Hodge structure on Y. insert precise references.

9.3. Kato-Usui spaces.

We fix $n, h, H_{\mathbb{Z}}, Q, G_{\mathbb{Z}}, D$ and $D^\vee$ as in (9.1.1). Let $\mathfrak{g}_R = \mathrm{Lie}(G_R)$. A subset $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ is called a *nilpotent cone* if it is a cone

$$\sigma = \sum_{i=1}^{n} \mathbb{R}_{\geq 0} N_i$$

generated by mutually commutative nilpotent operators $N_i \in \mathfrak{g}_{\mathbb{R}} \subset \mathrm{End}(H_{\mathbb{R}})$. Let Γ be a *neat subgroup* of $G_{\mathbb{Z}}(\mathbb{Z})$, i.e. for every element $\gamma \in \Gamma$, its eigenvalues on $H_{\mathbb{C}}$ generate a torsion-free subgroup of $\mathbb{C}^*$.

9.3.1. Nilpotent orbits.

nilp-orbit

Definition 9.3. Let $\sigma = \sum_i \mathbb{R}_{\geq 0} N_i$ be a nilpotent cone. A subset $Z \subset D^\vee$ is called a *σ-nilpotent orbit*, if there exists an $F_0 \in D^\vee$ such that

- $Z = \exp(\sum_i \mathbb{C}N_i)F_0$,
- $NF_0^p \subset F_0^{p-1}$ for all $p \in \mathbb{Z}$ and $N \in \sigma$,
- $\exp(\sum_i z_i N_i)F_0 \in D$ if $Im(z_i) \gg 0$ for all i.

We also call the pair (σ, Z) a *nilpotent orbit*.

Let Σ be a *fan* in $\mathfrak{g}_{\mathbb{Q}}$, i.e. $\Sigma \neq \emptyset$ is a set of rational nilpotent cones in $\mathfrak{g}_{\mathbb{R}}$ (namely, those generated by nilpotent operators in $\mathfrak{g}_{\mathbb{Q}}$) such that

- if $\sigma \in \Sigma$, then all faces of σ are in Σ,
- for $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ',

- for every $\sigma \in \Sigma$, we have $\sigma \cap (-\sigma) = 0$.

One can then define the set $D_{h,\Sigma}$ (or just D_Σ, if there is no confusion) of *nilpotent orbits in the directions in* Σ to be the set of nilpotent orbits (σ, Z) where $\sigma \in \Sigma$. There is a natural injection

$$F \mapsto (0, \{F\}) : D \hookrightarrow D_\Sigma.$$

9.3.2. The moduli space M_Σ. Let Σ be a fan in $\mathfrak{g}_\mathbb{Q}$ and let $\Gamma \subset G_\mathbb{Z}(\mathbb{Z})$ be a subgroup. Then we say that Γ *is compatible with* Σ if for every $\gamma \in \Gamma$ and $\sigma \in \Sigma$, we have $Ad(\gamma)(\sigma) \in \Sigma$. In this case, there is an action of Γ on D_Σ given by

$$(\sigma, Z) \xmapsto{\gamma} (Ad(\gamma)(\sigma), \gamma Z).$$

We say that Γ *is strongly compatible with* Σ if every cone $\sigma \in \Sigma$ is generated by elements in $\log \Gamma$. Kato and Usui showed that when Γ is strongly compatible with Σ and the arithmetic subgroup Γ is neat, the quotient set $\Gamma \backslash D_\Sigma$ can be given the structure of a log locally ringed space over $\mathbb{C}$, in fact a *log manifold* (see ([20], 3.5.7)). Roughly speaking, a log manifold is a log locally ringed space over $\mathbb{C}$, which is locally isomorphic to the "zero locus" of some log differential forms on a log smooth analytic space.

mention that not a complex manifold and give examples

Informally speaking, Kato and Usui proved the following. First, there is a one-to-one correspondence between D_Σ and the set of polarized log Hodge structures of the given type. Second, if $\overline{X} \to \overline{S}$ is a log smooth family extending the projective smooth family $X \to S$, where $S \subset \overline{S}$ is the complement of a normal crossing divisor, then the period map extends to $\overline{S} \to M_\Sigma$. We briefly explain the first part in the following.

We shall show how to get a nilpotent orbit from a polarized log Hodge structure on a log point ([20], 0.4.24). Let x be an fs log point with log structure M_x. Then $\overline{M}_x$ is a sharp fs monoid and $\overline{M}_x^{\mathrm{gp}}$ if a free abelian group of finite rank, say r. Fix $y \in x^{\log}$. We have $x^{\log} = Hom(\overline{M}_x^{\mathrm{gp}}, S^1) \simeq (S^1)^r$ and hence $\pi_1(x^{\log}) = Hom(\overline{M}_x^{\mathrm{gp}}, \mathbb{Z}) \simeq \mathbb{Z}^r$. Let $\pi_1^+(x^{\log}) \subset \pi_1(x^{\log})$ be the subset consisting of those homomorphisms $a : \overline{M}_x^{\mathrm{gp}} \to \mathbb{Z}$ that take $\overline{M}_x$ into $\mathbb{N}$; this subset is an fs monoid.

Let $(H_\mathbb{Z}, Q, F^\bullet H_\mathscr{O})$ be a polarized log Hodge structure on x. Let $(h_i)_{i=1}^n$ be a family of generators for $\pi_1^+(x^{\log})$ and $s_0 \in \mathrm{sp}(y)$. Let $z_1, \cdots, z_r$ be complex numbers, and let $s \in \mathrm{sp}(y)$ be such that

$$s\Big(\frac{\log(f)}{2\pi i}\Big) - s_0\Big(\frac{\log(f)}{2\pi i}\Big) = \sum_{i=1}^{r} z_i h_i(f), \quad \text{for } f \in \overline{M}_x^{\mathrm{gp}}.$$

Let $N_i : H_{\mathbb{Q},y} \to H_{\mathbb{Q},y}$ be the logarithm of h_i. Then we have

$$F(s) = \exp\Big(\sum_{i=1}^{n} z_i N_i\Big) F(s_0),$$

which shows that $(F(s))_{s\in\mathrm{sp}(y)}$ is an orbit of filtrations under $\exp(\sigma\otimes\mathbb{C})$ for $\sigma=\sum_i \mathbb{R}_{\geq 0}N_i$. Moreover, the condition 2) in (9.2) implies that $F(s)\in D$ if $\mathrm{Im}(z_i)\gg 0$ for all i, and the condition 3) in (9.2) implies that $NF(s_0)^p\subset F(s_0)^{p-1}$ for all $p\in\mathbb{Z}$ and $N\in\sigma$. In other words, the family $(F(s))_s$ is a σ-nilpotent orbit.

MainComp

10. The main component of moduli spaces

10.1. Moduli: compactness and main components

In Section 4, we gave an overview of F. Kato's work [17] in which he uses log geometry to compactify the moduli space $\mathcal{M}_{g,n}$ of curves. Specifically, he shows that the moduli space of log smooth curves agrees with the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$. The key philosophic idea in that section was that since moduli spaces of log smooth objects already includes degenerate objects, it is reasonable to expect that such a moduli space is a compactification of the moduli of objects with trivial log structure.

While the Deligne–Mumford space of stable curves $\overline{\mathcal{M}}_{g,n}$ turns out to be irreducible, it is an unfortunate fact of life that if $\mathfrak{X}$ is a moduli space of higher dimensional objects, then moduli-theoretic "compactifications" $\bar{\mathfrak{X}}$ of $\mathfrak{X}$ tend to have many irreducible components. If $\mathfrak{X}$ is irreducible, then it sits entirely within one of these many components of $\bar{\mathfrak{X}}$ and so it is natural then to ask if this "main component" can itself be given a moduli interpretation.

In Section 4.4 we stated a second philosophic principle: log geometry controls degenerations; that is, moduli of log smooth objects does not incorporate "too many" degenerate objects. This provides a type of converse to the aforementioned philosophy that moduli of log smooth objects should be compact. Combining the two principles, one would expect that moduli of log smooth objects will not only be compact, it will also be *irreducible.* In particular, one would imagine that if $\mathfrak{X}$ is an irreducible moduli space and $\bar{\mathfrak{X}}$ a moduli-theoretic compactification, then by appropriately incorporating log structures into the objects parameterized by $\bar{\mathfrak{X}}$, one will isolate the main component.

I don't see any reason why moduli of log smooth schemes should be irred. in general

This technique of using log geometry to isolate the main component of a moduli space has been carried out by one of us (M. Olsson) in several different settings. In [31], Olsson gives a moduli interpretation to the normalization of the main component of the toric Hilbert scheme. In [38], he isolates the normalization of the main component of V. Alexeev's compactification of the moduli space of principally polarized abelian varieties given in [3]; he further constructs a moduli-theoretic irreducible compactification of the moduli space of abelian varieties with higher degree polarization. In [34], he gives an irreducible modular compactification of the moduli space of polarized K3 surfaces.

it seems strange to refer to one of the authors in this way.

10.2. Example: the toric Hilbert scheme

Our goal in this section is to explain the technique of isolating the main component of a moduli space by following Olsson's work [31]. We begin with the definition of the toric Hilbert scheme. Let k be a field and let P and Q be finitely-generated integral monoids with Q sharp and P^{gp} and Q^{gp} torsion-free. Fix a surjective morphism $\pi : P \to Q$. This yields a closed immersion from $A_Q := \operatorname{Spec} k[Q]$ to $A_P := \operatorname{Spec} k[Q]$, which is T_Q-equivariant, where T_Q (resp. T_P) denotes the torus associated to Q^{gp} (resp. P^{gp}). Consider the functor $\mathcal{H}$ whose S-valued points are diagrams

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[dr, "g"'] & A_{P,S} \arrow[d] \\ & S \end{tikzcd}$$

where i is a T_Q-invariant closed immersion and for every $q \in Q^{gp}$, the q-eigenspace of $g_*\mathcal{O}_Z$ is a finitely-presented projective $\mathcal{O}_S$-module of rank 1 if $q \in Q$ and rank 0 otherwise. By [12, Thm 1.1], this functor is representable by a quasi-projective scheme, which we call the *toric Hilbert scheme.*

Given a closed subscheme Z of $A_{P,S}$ as above, we can move Z by the action of T_P on $A_{P,S}$. This yields an action of T_P on $\mathcal{H}$. Since Z is T_Q-invariant, this action factors through $T_K = T_P/T_Q$, where K denotes the kernel of π^{gp}. We therefore ~~we~~ obtain a map

$$T_K \longrightarrow \mathcal{H}$$

by letting $u \in T_K$ act on the distinguished point of $\mathcal{H}(k)$ given by the closed immersion $A_Q \to A_P$. By [6, 3.6(2)], this map is an open immersion. Therefore the normalization $\mathcal{S}$ of the scheme-theoretic closure of its image is a normal toric variety, and hence carries a natural fs log structure $\mathcal{M}_{\mathcal{S}}$ which makes it log smooth over $\operatorname{Spec} k$ (endowed with the trivial log structure). The goal of [31] is to give a moduli-theoretic interpretation of $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$.

Consider the functor $\mathcal{H}^{log}$ on the category of fs log schemes over k whose $(S, \mathcal{M}_S)$-valued points are given by diagrams

$$\begin{tikzcd} (Z, \mathcal{M}_Z) \arrow[r, "i"] \arrow[dr, "g"'] & (A_P, \mathcal{M}_{A_P}) \times (S, \mathcal{M}_S) \arrow[d] \\ & (S, \mathcal{M}_S) \end{tikzcd}$$

where the underlying maps on schemes defines a point of $\mathcal{H}(S)$, where g is log smooth and integral, and where the map

$$P \longrightarrow \mathcal{M}_{(Z,\mathcal{M}_Z)/(S,\mathcal{M}_S)} := \operatorname{coker}(g^*\mathcal{M}_S \to \mathcal{M}_Z)$$

induced by i factors through Q. The main theorem of [31] is then thm:mainthm **Theorem 10.1** ([31, Thm 1.6]). *The functor $\mathcal{H}^{log}$ is representable by $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$.*

Is the rest of this dimension nec. to include?

The first step in proving this theorem is to obtain a map from the functor points $\mathcal{H}_{\mathcal{S}}$ of $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$ to $\mathcal{H}^{log}$. This is done by showing that the pullback to $\mathcal{S}$ of the universal family over $\mathcal{H}$ yields a point of $\mathcal{H}^{log}(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$. Explicitly, if $i : \mathcal{Z} \to A_P \times \mathcal{S}$ is the pullback of the universal family, Olsson constructs a log structure on $\mathcal{Z}$ as follows. Since $\mathcal{S}$ is a toric variety with torus T_K, it can be covered by open affines of the form $\operatorname{Spec} k[L]$ with L a submonoid of K whose associated group is K. Olsson proves ([31, 2.4]) that over such an open affine, the closed immersion i is given by

$$\mathcal{Z} \times_{\mathcal{S}} \operatorname{Spec} k[L] = \operatorname{Spec} k[E_L] \longrightarrow \operatorname{Spec} k[P \oplus L] = A_P \times \operatorname{Spec} k[L],$$

where E_L is the image of $P \oplus L$ in P^{gp} under the map $(p, \ell) \mapsto p + \ell$.

Example 10.2. *Let $\pi : \mathbb{N}^2 \to \mathbb{N}$ send e_1 to 2 and e_2 to 1. Then the kernel K of π^{gp} is all integer multiples of $2e_2 - e_1$. Let L be the submonoid of K consisting of the non-negative multiples of $e_1 - 2e_2$ and let M be the submonoid consiting of the non-positive multiples. Then E_L is generated by e_1, e_2, and $2e_2 - e_1$; hence,*

$$\operatorname{Spec} k[E_L] \simeq k[x, y, z]/(xy - z^2).$$

We see that E_M is freely generated by e_2 and $e_1 - 2e_2$, so $\operatorname{Spec} k[E_M] \simeq \mathbb{A}^2_k$.

We see then that $\mathcal{Z} \times_{\mathcal{S}} \operatorname{Spec} k[L]$ carries a natural log structure. These log structures glue to give a log structure on $\mathcal{Z}$ (by [31, Lemma 2.8]) which makes i a closed immersion of log schemes. We therefore obtain a morphism of functors

$$F : \mathcal{H}_{\mathcal{S}} \longrightarrow \mathcal{H}^{log}.$$

The next step in proving Theorem 10.1 above is to work out the deformation theory of $\mathcal{H}^{log}$. Olsson shows ([31, §4]) that if $(\operatorname{Spec} B_0, \mathcal{M}_{B_0}) \to (\operatorname{Spec} B, \mathcal{M}_B)$ is a strict closed immersion defined by a square zero ideal $\mathcal{I}$, then every object of $\mathcal{H}^{log}(\operatorname{Spec} B_0, \mathcal{M}_{B_0})$ lifts to an object of $\mathcal{H}^{log}(\operatorname{Spec} B, \mathcal{M}_B)$; moreover, the set of lifts forms a torsor under $\operatorname{Hom}(K, \mathcal{I})$. Since $\mathcal{S}$ is a toric variety with torus T_K, we see

$$\Omega^1_{(\mathcal{S}, \mathcal{M}_{\mathcal{S}})/k} = \mathcal{O}_{\mathcal{S}} \otimes_{\mathbb{Z}} K.$$

As a result, every object of $\mathcal{H}_{\mathcal{S}}(\operatorname{Spec} B_0, \mathcal{M}_{B_0})$ also lifts to an object of $\mathcal{H}_{\mathcal{S}}(\operatorname{Spec} B, \mathcal{M}_B)$ and the set of lifts forms a torsor under $\operatorname{Hom}(K, \mathcal{I})$. This shows

thm:loget **Theorem 10.3** ([31, 4.4]). *The functor F is formally log étale.*

Olsson then makes use of Theorem 10.3 to show that F is injective and surjective, thereby proving Theorem 10.1. We explain his proof of injectivity as it gives a concrete instance of the philosophy and diagram of Section 4.4. Suppose f and g are two morphisms from an fs log scheme $(T, \mathcal{M}_T)$ to $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$ such that $Ff = Fg$. We show that $f = g$. Since $\mathcal{S}$ is locally of finite presentation over k, a limit argument allows us to assume T is of finite type over k. Since $\mathcal{H}^{log}$ is limit preserving (by [31, §6]), another limit argument shows that it is enough to check $f = g$ at the completion of T at every point. Since T is assumed to be of finite type over k (hence Noetherian), we have reduced to the case $T = \operatorname{Spec} R$ for R a complete local Noetherian ring. To check $f = g$, it now suffices to show that this is true after precomposing with all maps $\operatorname{Spec} R/\mathfrak{m}^n \to \operatorname{Spec} R$, where $\mathfrak{m}$ is the maximal ideal of R. Hence, we may assume $T = \operatorname{Spec} B$, where B is a local Artin ring. If $i : (\operatorname{Spec} B_0, \mathcal{M}_{B_0}) \to (\operatorname{Spec} B, \mathcal{M}_B)$ is a strict closed immersion with square zero ideal, and if $fi = gi$, then the diagram

$$\begin{tikzcd}
(\operatorname{Spec} B_0, \mathcal{M}_{B_0}) \arrow[r, "fi=gi"] \arrow[d, "i"'] & (\mathcal{S}, \mathcal{M}_{\mathcal{S}}) \arrow[d, "F"] \\
(\operatorname{Spec} B, \mathcal{M}_B) \arrow[r] \arrow[ur, shift left, "f"] \arrow[ur, shift right, "g"'] & \mathcal{H}^{log}
\end{tikzcd}$$

commutes. By Theorem 10.3, we see then that $f = g$. We can therefore assume that $T = \operatorname{Spec} k'$ with k' a field. It further suffices to assume that k' is separably closed, in which case there is an fs sharp monoid L and a morphism $L \to \mathcal{M}_T$ inducing an isomorphism of L with $\bar{\mathcal{M}}_T$.

Consider the complete local ring $R' = k'[[L]]$ defined as the completion of $k'[L]$ at the maximal ideal $\mathfrak{m}'$ which is the kernel of the morphism $k'[L] \to k'$ sending all non-zero elements of L to 0. The scheme $\operatorname{Spec} R'$ carries a natural log structure $\mathcal{M}_{R'}$ defined as the pullback of the canonical log structure on $\operatorname{Spec} \mathbb{Z}[L]$. By Theorem 10.3 and log smoothness of $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$, we obtain morphisms $\tilde{f}$ and $\tilde{g}$ from $(\operatorname{Spec} R', \mathcal{M}_{R'})$ to $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$ such that $F\tilde{f}_n = F\tilde{g}_n$ for every n, where $\tilde{f}_n$ (resp. $\tilde{g}_n$) denotes the reduction of $\tilde{f}$ (resp. $\tilde{g}$) modulo $\mathfrak{m}'^n$.

In other words, we have replaced our original T, f, and g by $\operatorname{Spec} R'$, $\tilde{f}$, and $\tilde{g}$. So, we are once again in the case where T is the spectrum of a complete local Noetherian ring. What have we gained, then, over the previous case when $T = \operatorname{Spec} R$ above? We have succeeded in replacing an arbitrary complete local Noetherian ring R by a very special one, namely $k'[[L]]$. We saw these special rings come up in the diagram of Section 4.4.

Let $h : \mathcal{S} \to \mathcal{H}$ be the natural morphism (recall that $\mathcal{S}$ is the normalization of the main component of $\mathcal{H}$). Since $F\tilde{f}_n = F\tilde{g}_n$ for all n, the very definition of $\mathcal{H}^{log}$ shows that $f\tilde{f}_n = h\tilde{g}_n$ for all n. Therefore, $h\tilde{f} = h\tilde{g}$. Since $k'[[L]]$ is normal and $\mathcal{S}$ is separated, to show that $\tilde{f} = \tilde{g}$, it suffices to show they are equal on a dense open subset. Since the generic point of $k'[[L]]$ has trivial log structure, it factors through the torus T_K of $\mathcal{S}$. The map h, however, is an open immersion over T_K, and so $\tilde{f}$ and $\tilde{g}$ agree on the generic point of $k'[[L]]$.

would be nice to discuss here the case of K3 surfaces?

Roots

11. Twisted curves and log twisted curves

Twisted curves are a central object in the theory of *twisted stable maps* [2, 5, 1]: in order to have a complete moduli space of stable maps $C \to X$ of type

Γ, where X is a proper tame stack with projective coarse moduli space and $\Gamma = (g, n, \beta)$ are the relevant discrete data, one must allow the curve C itself to be a certain type of stack, called *twisted curve.*

The original treatments of twisted curves relied on ad-hoc methods. The more recent approach of [1] relies on a method introduced in [37], which uses a construction with logarithmic structures.

11.1. Twisted curves

For simplicity we will stick with the case of Deligne–Mumford stacks.

First consider the geometric objects: fix an algebraically closed field k.

Definition 11.1. A *twisted curve* over k is a tame, purely 1-dimensional Deligne–Mumford stack $\mathcal{C}/k$, with at most nodes as singularities, satisfying the following conditions:

numbering.

(1) Let $\pi : \mathcal{C} \to C$ be the morphism to the coarse moduli space. Then $\mathcal{C}^{sm} = \pi^{-1}C^{sm}$, and $\pi : \mathcal{C} \to C$ is an isomorphism over a dense open subset of C.

(2) Consider a node $\bar{x} \to C$, where the strictly henselian local ring $\mathcal{O}_{C,\bar{x}}$ is the strict henselization of $k[x, y]/(xy)$. Then

$$\mathcal{C} \times_C \operatorname{Spec} \mathcal{O}_{C,\bar{x}} \simeq \left[\operatorname{Spec} \mathcal{O}_{C,\bar{x}}[z, w]/(zw, z^m - x, w^m - y) \Big/ \mu_m\right],$$

where $\zeta \in \mu_m$ acts by $(z, w) \mapsto (\zeta Z, \zeta^{-1}w.)$

An action such as (2) above is called *balanced.* Note that $\mathcal{C}$ may have a stack structure at isolated smooth points as well - such points will behave like $[\mathbb{A}^1/\mu_a]$.

where μ_a action is multiplication

Over a general base S twisted curves are detected by their geometric fibers: a twisted curve $\mathcal{C} \to S$ is a flat, tame Deligne–Mumford stack locally of finite presentation, all of whose geometric fibers are twisted curves as in the definition above.

The *genus* of $\mathcal{C}$ is simply the genus of C. One typically needs to consider n-pointed twisted curves, where the markings are described in families as follows:

Definition 11.2. An *n-pointed twisted curve* $\mathcal{C}/S$ marked by disjoint closed substacks $\{\Sigma_i\}_{i=1}^n$ in $\mathcal{C}$ is assumed to satisfy the following:

(1) the Σ_i are contained in the smooth locus $\mathcal{C}^{sm}$,
(2) each Σ_i is a tame étale gerbe over S, and
(3) $\mathcal{C}_{gen} := \mathcal{C}^{sm} \smallsetminus \cup_i \Sigma_i \longrightarrow C$ is an open embedding.

Remark 11.3. When $S = \operatorname{Spec} k$ where $k = \bar{k}$, then $\Sigma_i = B\mu_{a_i}$, and moreover a_i is locally constant in families.

Remark 11.4. When $(\mathcal{C}/S, \{\Sigma_i\})$ is an n-pointed twisted curve, then the coarse moduli space of Σ_i is isomorphic to S. This means that the composite morphism

$\Sigma_i \to \mathcal{C} \to C$ factors through a section $p_i : S \to C$. It follows that $(C, \{p_i\})$ is an n-pointed curve in the usual sense. This gives a functor

$$(\mathcal{C}/S, \{\Sigma_i\}) \mapsto (C, \{p_i\})$$

One can ask oneself: what does one need in order to recover a twisted n-pointed curve $(\mathcal{C}/S, \{\Sigma_i\})$ from a usual n-pointed curve $(C, \{p_i\})$? In other words, can we enrich the functor above to something like

$$(\mathcal{C}/S, \{\Sigma_i\}) \mapsto (C, \{p_i\}) + ?$$

which is nice and explicit and actually an equivalence of categories?

The stack structure at the marking definitely needs the data of the integers a_i, but in fact this is all that is necessary for the markings: near p_i, the curve $\mathcal{C}$ is canonically isomorphic to the root stack $C(\sqrt[a_i]{p_i})$. If x is a local generator of the ideal of p_i, then Zariski locally we have

$$\mathcal{C} \;\simeq\; \left[\operatorname{Spec} \mathcal{O}_C[z]/(z^{a_i} - x) \;\Big/\; \mu_{a_i}\right].$$

The story is a bit more interesting at a node. It has to be - a twisted curve $\mathcal{C}$ with a node of index $m > 1$ has "ghost" automorphisms in μ_m which are not detectible on the coarse curve C.

write out more explicitly

11.2. Log twisted curves

Let X be a Deligne–Mumford stack. Recall that a fine log structure M on X is said to be *locally free* if for every geometric point $\bar{x} \to X$ we have that the characteristic sheaf $\overline{M}_{\bar{x}}$ is isomorphic to $\mathbb{N}^r$ for some r.

was this defined earlier

In this situation we say that a morphism of sheaves of monoids $M \to M'$ is *simple* if for every geometric point $\bar{x} \to X$ we can identify the map as the diagonal map

$$\begin{tikzcd}
\overline{M}_{\bar{x}} \arrow[r] \arrow[d, "\simeq"] & \overline{M}'_{\bar{x}} \arrow[d, "\simeq"] \\
\mathbb{N}^r \arrow[r, "(m_1,\ldots,m_r)"] & \mathbb{N}^r
\end{tikzcd}$$

where all a_i are prime to the characteristic of the field.

Definition 11.5. An *n-pointed log twisted curve over S* is the data

$$(C/S, \{\sigma_i, a_i\}, \ell : M_S \to M'_S)$$

where

- $(C, \{\sigma_i\})/S$ is an n-pointed nodal curve.
- M_S is the canonical log structure coming from the family $(C, M_C) \to (S, M_S)$ (see section 4).
- $a_i : S \to \mathbb{Z}_{>0}$ are locally constant, with $a_i(s)$ invertible in the residue field $k(s)$.

- $\ell : M_S \to M'_S$ is a simple morphism.

And we have the following:

ref.

Theorem 11.6. ✓ *The fibered category of n-pointed twisted curves is naturally equivalent to the stack of n-pointed log twisted curves.*

The picture is as follows: we have already noted that we can replace a marking p_i by a stacky marking just using the data a_i. Now the j-th node which looks like $\operatorname{Spec} \mathcal{O}_S[x,y]/(xy-t)$ needs to be replaced by $\left[\operatorname{Spec} \mathcal{O}_S[z,w]/(zw - t^{1/m_j}) \;/\; \mu_{m_j}\right]$, and the data is encoding by deviding the j-th generator of $\mathbb{N}^r$ by m_j.[5]

Remark 11.7. We can decompose the stack according to a_i:

$$\begin{array}{ccccl} \coprod_{\underline{a}} \mathcal{M}^{tw}_{g,n,\underline{a}} & = & \mathcal{M}^{tw}_{g,n} & = & \text{stack of } n\text{-pointed twisted curves} \\ & & \big\downarrow & & \\ & & \mathcal{S}_{g,n} & = & \text{stack of } n\text{-pointed nodal curves} \end{array}$$

and it can be deduced form the theorem that $\mathcal{M}^{tw}_{g,n,\underline{a}}$ is obtained from $\mathcal{S}_{g,n}$ using a root construction applied to the boundary divisor of $\mathcal{S}_{g,n}$. (ref)

In fact we have to apply all possible roots, accounting for all possible twisting of nodes, and glue together, so $\mathcal{M}^{tw}_{g,n,\underline{a}}$ is highly non-separated.

Below we sketch the main ideas in proving this.

11.3. From twisted curves to log twisted curves

Fix a twisted curve $f : \mathcal{C} \to S$.

We can follow F. Kato [17], giving log structures on nodal curves: consider all possible triples

$$(M_S, M_{\mathcal{C}}, f^\flat : f^{-1}M_S \to M_{\mathcal{C}})$$

such that

(1) $(\mathcal{C}, M_{\mathcal{C}}) \to (S, M_S)$ is log smooth;
(2) $M_{\mathcal{C}}, M_S$ are locally free; and
(3) for all geometric points $\bar{x} \to \mathcal{C}$ mapping to nodes we have

$$\begin{tikzcd} \overline{M}_{\mathcal{C},\bar{x}} \arrow[r, "\sim"] & \mathbb{N}^{r-1} \oplus \mathbb{N}^2 \\ \overline{M}_{S,f(\bar{x})} \arrow[u] \arrow[r, "\sim"] & \mathbb{N}^{r-1} \oplus \mathbb{N} \arrow[u, "id \oplus \Delta"'] \end{tikzcd}$$

[5] put here Martin's picture!!

If $S = \operatorname{Spec} k$ we get a natural map

$$\mathbb{N}^{\text{number of nodes}} \to \overline{M}_S.$$

We say that (M_C, M_S, f^b) is special if for every geometric point $\bar{s} \to S$ this map is an isomorphism. where?

A proposition analogous to Kato's says that there is a unique special triple $(M_{\mathcal{C}}, M'_S, f^b)$ associated to $f : \mathcal{C} \to S$. Analyzing the coarse moduli space we obtain a unique diagram

$$\begin{tikzcd}
(\mathcal{C}, M_{\mathcal{C}}) \arrow[r] \arrow[d] & (C, M_C) \arrow[d] \\
(S, M'_S) \arrow[r, "(id,\ell)"] & (S, M_S)
\end{tikzcd}$$

where $\ell : M_S \to M'_S$ is simple. In particular we obtain a log twisted curve

$$(C/S, \{\sigma_i, a_i\}_{i=1}^n, \ell : M_S \to M'_S)$$

11.4. From log twisted curves to twisted curves

Now we fix a log twisted curve $(C/S, \{\sigma_i, a_i\}_{i=1}^n, \ell : M_S \to M'_S)$. In particular we have the log smooth curve

$$(C, M_C) \to (S, M_S)$$

which is the coarse moduli space of a putative twisted curve. We want to describe $\mathcal{C}/C$ as the stack parametrizing natural objects over $T \to C$. Here it is! If we denote the relevant maps as follows

$$\begin{tikzcd}
T \arrow[r, "s"] \arrow[dr, "h"'] & C \arrow[d] \\
 & S,
\end{tikzcd}$$

then $\mathcal{C}$ is the groupoid of diagrams

$$\begin{tikzcd}
h^* M_S \arrow[r, "\ell"] \arrow[d] & H^* M'_S \arrow[d] \\
s^* M_C \arrow[r, "k"] & M'_C,
\end{tikzcd}$$

where

(1) k is simple and for every geometric point $\bar{t} \to T$ the map $\overline{M}'_{S,\bar{t}} \to \overline{M}'_{C,\bar{t}}$ is either an isomorphism (at a general point), or of the form

$$\mathbb{N}^r \xhookrightarrow{(id,0)} \mathbb{N}^r \oplus \mathbb{N}$$

(at a marked point), or

$$\mathbb{N}^{r-1} \oplus \mathbb{N} \xhookrightarrow{(id,\Delta)} \mathbb{N}^{r-1} \oplus \mathbb{N}^2$$

(at a node).

(2) for all i and geometric point $\bar{t} \to T$ with $s(\bar{t}) \subset \sigma_i(S) \subset C$ the group

$$\mathrm{Coker}(\overline{M}^{gp}_{C,\bar{t}} \to \overline{M'}^{gp}_{C,\bar{t}})$$

is cyclic of order a_i.

Remark: this does not work for unbalanced curves!

Stablemaps

12. Log stable maps

12.1. From curves to maps and expansions

12.1.1. Curves In section 4 we discussed how prestable curves can be encoded as log smooth curves, and how in particular the stack of Deligne–Mumford stable curves can be interpreted as a logarithmic stack, representing log smooth stable curves over the category of fine and saturated log schemes. Stability in this situation just means that $\Omega^1_{C^{\log}}$ is an ample line bundle. This is the same as saying that $\omega_C(D)$ is ample, where D is the divisor of markings.

compare with earlier notation

12.1.2. Maps Kontsevich [23] introduced the moduli of *stable maps* of prestable curves into a projective target variety X. This is a proper Deligne–Mumford stack $\overline{\mathcal{M}}_\Gamma(X)$ having projective coarse moduli space, where $\Gamma = (g, n, \beta)$ is the relevant numerical data: genus, number of markings and homology class of the image curve. It parametrizes maps $f : C \to X$, where this time stability means that ω_C is f-ample.

Kontsevich's moduli space has the property that it carries a *perfect relative obstruction theory* giving rise to a *virtual fundamental class* $[\overline{\mathcal{M}}_\Gamma(X)]^{vir}$, see [27, 4]. This is a key ingredient in defining Gromov–Witten invariants, with their applications in enumerative geometry and theoretical physics. The simplest GW invariants are

$$\langle \gamma_1 \cdots \gamma_n \rangle := \int_{[\overline{\mathcal{M}}_\Gamma(X)]^{vir}} e_1^*\gamma_1 \cdots e_n^*\gamma_n,$$

where $\gamma_i \in H^*(X, \mathbb{Q})$ and $e_i : \overline{\mathcal{M}}_\Gamma(X) \to X$ are the natural *evaluation maps* at the n markings.

So far no logarithmic structures are necessary.

12.1.3. Degenerations Among the methods of computing GW invariants, the *degeneration formula* is among the most powerful ones. It was introduced by A.-M. Li and Y. Ruan in symplectic geometry [24], see also Ionel-Parker [14, 15]. For our purposes, Jun Li's treatment in algebraic geometry [25, 26] is most relevant.

We are interested in the invariants of the smooth projective variety X. Since these are deformation invariant, it is natural to consider a degeneration of X, with smooth total space, into a union $X_0 = Y_1 \sqcup_D Y_2$ of two smooth varieties Y_i meeting transversally along a smooth divisor D. One wants to show that invariants of X coincide with suitably defined invariants of X_0, and these in turn can be computed in terms of appropriately defined invariants of Y_i relative to D.

This is where logarithmic structures begin to show up, but there is still some way to go.

12.1.4. Perfect obstruction and Li's approach The difficulty with the degeneration is precisely the fact that the variety X_0 is singular, and therefore the natural obstruction theory on the moduli space of stable maps is not perfect in general. A similar situation occurs when considering the pair (Y_i, D), but we will not get into this discussion.

The problem occurs when a component of the source curve C maps entirely into D.

Jun Li's approach uses *expanded degenerations.* There are similar ideas in [24], but the symplectic approach builds in deformations of Cauchy-Riemann equations and has, at least on the surface, a significantly different flavor.

The idea is, that just as in stable pointed curves, if a marking travels towards a node one sprouts a new component of the curve, Jun Li says that when a component of C travels into D we can let X_0 sprout a new component.

12.1.5. Expansions Here is how the new component looks like. Denote by $N_{D\subset Y_i}$ the normal bundle of D in Y_i. Since the total space is smooth, we have $N_{D\subset Y_1} \cong N^\vee_{D\subset Y_2}$. Let $\mathbb{P} = \mathbb{P}roj_D(1_D \oplus N_{D\subset Y_1})$. We have $\mathbb{P} \cong \mathbb{P}roj_D(N_{D\subset Y_2} \oplus 1_D)$ so we can denote by D^+ and D^- the smooth divisors in $\mathbb{P}$ which correspond to the normal bundle $N_{D\subset Y_1}$ and $N_{D\subset Y_2}$ respectively. Note that the divisor D^+ and D^- are canonically isomorphic to D.

We now glue things up. Let $\mathbb{P}_i$ for $i \in \mathbb{N}$ be copies of $\mathbb{P}$. We can glue Y_1 and $\mathbb{P}_1$ along D and D^- respectively, $\mathbb{P}_i$ and $\mathbb{P}_{i+1}$ along D^+ and D^- respectively, and $\mathbb{P}_n$ and Y_2 along D^+ and D respectively. We denote the resulting gluing by

$$X_0[n] = Y_1 \bigsqcup_{D_1} \mathbb{P}_1 \bigsqcup_{D_2} \cdots \bigsqcup_{D_n} \mathbb{P}_n \bigsqcup_{D_{n+1}} Y_2, \tag{12.1}$$

where $D_1, \cdots, D_{n+1}$ are the disjoint singular loci of $X_0[n]$.

Such a beast is known as an *expanded degeneration*, or by the more folksy name *an n-accordion.*

12.1.6. Predeformability So Jun Li comes to define *degeneration stable maps with target X_0 as nondegenerate maps $C \to X_0[n]$:* a map is nondegenerate if no component of C maps into any of the D_i.

But here Jun Li meets another phenomenon, already present in the space of admissible covers of Harris–Mumford [13]: nondegenerate maps to $C \to X_0[n]$ have many redundant components which have nothing to do with maps to the generic fiber X. Near a singular point of $X_0[n]$ which looks like $\{xy = 0\}$, a curve map locally deforms to a smoothing of X_0 if and only if the curve looks like $\{uv = 0\}$, and the map given by $x = u^m, y = v^m$. Such nice maps are called *predeformable.* But predeformable maps are clearly not open in the space of maps - they are actually closed among nondegenerate maps. This means that the restricted obstruction theory on them is "wrong" - definitely not perfect

Of course the virtual fundamental class of Gromov–Witten theory has no problem dealing with the total moduli space with its extra components, but these extra components do get in the way of decomposing invariants of X_0 in terms of (Y_i, D). So we really do want to stick by predeformable maps.

12.2. Logarithmic methods: from Jun Li to Bumsig Kim

12.2.1. Predeformable deformations So Jun Li comes to need to a new obstruction theory on predeformable maps. Having read this chapter, the reader will immediately recognize that — wording

(1) nodal curves are log-smooth,
(2) n-accordions are d-semistable and admit a canonical log-smooth structure, and
(3) predeformble maps can be viewed as log maps from the log-smooth curve to the d-semistable target.

Jun Li also recognized this fact, as did Shin Mochizuki before him [29] when he in his turn revisited the space of admissible covers of Harris–Mumford. What he lacked at the time was a formalism for logarithmic deformation theory of singular spaces, such as the moduli space itself: a perfect obstruction theory is a two-term complex mapping to the cotangent complex of the moduli space, but the moduli space is, as usual highly singular, even taking its logarithmic structure into account.

So Jun Li resorted to an ad-hoc construction of his perfect obstruction theory. This is the most difficult part of his work.

In section 7 we saw how log deformation theory works in the necessary generality. This is where Bumsig Kim's paper [21] comes in: he provides a correct formalism for nondegenerate logarithmic stable maps into expanded degenerations, and shows that it carries a perfect obstruction theory. The degeneration formula in this formalism has been worked out by one of us (Q. Chen), and should appear as part of a larget project indicated below.

One aspect that deserves mention is Kim's notion of *minimal log strutures* on maps. Recall that the log stack $\overline{\mathcal{M}}_{g,n}$ can be constructed as a stack over the category of fine saturated log schemes, whose objects over $S = (\underline{S}, M)$ are log smooth stable pointed curves over S. But in order to exhibit the underlying stack, one needs to use the *canonical* log structure on $\underline{S}$, which is initial among all possible ones carrying the log smooth curve.

Kim describes his stack similarly - given a predeformable map $\underline{C} \to \underline{X}$ of underlying schemes over $\underline{S}$, it amounts to describing what he calls *minimal* log structure S on $\underline{S}$ carrying a log map $C \to X$. Kim does this by explicitly describing the combinatorial structure of such log structures.

12.3. Unexpanded log maps: from Siebert into the future

A very different approach was proposed in a 2001 lecture by Bernd Siebert, but laid dormant for almost a decade.

The point is this. If one embraces logarithmic structures, and logarithmic maps from log smooth curves to some logarithmic scheme, then expansions are no longer necessary. Defined correctly, the space of log maps automatically has a perfect log-obstruction theory, which in view of sections 6 and 7 can be viewed as an obstruction theory relative to the stack LOG. This automatically results in a virtual fundamental class.

With this way of thinking, one can consider much more general logarithmic stable maps, gaining access to invariants of much more general degenerations of varieties. This has been a desired goal for a number of years.

So what is the correct definition? Consider a fine and saturated log scheme X. Following the work of F. Kato [17] as discussed in section 4, one comes up with a definition of a category $\overline{\mathcal{M}}_\Gamma(X)$ fibered over the category $LSch_{fs}$ of fine and saturated log schemes: an object over a fine saturated log scheme S is a log smooth curve $C \to S$ and a log map $f : C \to X$. We further require it to be *stable*: the line bundle of logarithmic differentials $\Omega^1_{C/S}$ is required to be f-ample. This is tantamount to requiring that the map of underlying schemes be a stable map.

So the main claim is: $\overline{\mathcal{M}}_\Gamma(X)$ is represented by a logarithmic Deligne–Mumford stack with projective coarse moduli space. In fact this stack is proper and quasi-finite over the usual stack of stable maps $\overline{\mathcal{M}}_\Gamma(\underline{X})$ of the underlying scheme. As in the discussion of Kim's work, the underlying stack can be viewed as a moduli of log maps with *minimal* log structure.

This is the subject of current work - of Gross and Siebert on the one hand and of two of us (mainly Chen, and to a lesser extent Abramovich) on the other, so it would not be appropriate to get into details until definite results appear.

Let us instead put this in a larger context. Consider logarithmic schemes $Z \to B$ and X, and assume we are given a morphism of underlying schemes $\underline{f} : \underline{Z} \to \underline{X}$. We can define a category $\mathrm{Lift}_{\underline{f}}$ fibered over $LSch_{fs}$ whose objects over a fine saturated log B-scheme $S \to B$ are lifts $f_S : Z_S \to X$ of the morphism of underlying schemes $\underline{f}_{\underline{S}} : \underline{Z}_{\underline{S}} \to \underline{X}$.

One can ask the following general questions:

Question 12.2. (1) Under what conditions is $\mathrm{Lift}_{\underline{f}}$ a log stack locally of finite type over B?

(2) What natural numerical data cut out a substack of finite type?

(3) Under what conditions is the result proper?

We want to stress our belief that this question is natural, important and quite tractable. For instance, the case where $B = \operatorname{Spec}\mathbb{C}$ with trivial structure and $Z = X = \operatorname{Spec}(\mathbb{N} \to \mathbb{C})$ the result is a countable union of components. It is similar in nature to an inertia stack. The more general case where $B = X$, $Z = X \times \operatorname{Spec}(\mathbb{N} \to \mathbb{C})$ and $\underline{f}$ is the diagonal, is the relevant analogue of the inertia stack of X. It is important for Gromov–Witten theory - up to $\mathbb{C}^*$ action the result is the natural target for evaluation maps associated to log smooth curves. Its components account for the contact orders of relative stable maps. Further examples of a similar nature govern gluing of nodes of log-smooth curves.

References

Olsson-Vistoli [1] Dan Abramovich, Martin Olsson, and Angelo Vistoli. Twisted stable maps to tame artin stacks, 2008. `http://arxiv.org/abs/0801.3040`. 40

movich-Vistoli [2] Dan Abramovich and Angelo Vistoli. Compactifying the space of stable maps. *J. Amer. Math. Soc.*, 15(1):27–75 (electronic), 2002. 40

alexeev [3] Valery Alexeev. Complete moduli in the presence of semiabelian group action. *Ann. of Math. (2)*, 155(3):611–708, 2002. 37

hrend-Fantechi [4] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997. 45

Chen-Ruan [5] Weimin Chen and Yongbin Ruan. Orbifold Gromov-Witten theory. In *Orbifolds in mathematics and physics (Madison, WI, 2001)*, volume 310 of *Contemp. Math.*, pages 25–85. Amer. Math. Soc., Providence, RI, 2002. 40

chuv [6] Olga V. Chuvashova. The main component of the toric Hilbert scheme. *Tohoku Math. J. (2)*, 60(3):365–382, 2008. 38

Del1 [7] Pierre Deligne. Travaux de griffiths. In *Séminaire Bourbaki. Vol. 1969/70: Exposés 364–381*, Lecture Notes in Mathematics, Vol. 180, pages 213–237: Exposé 376. Springer-Verlag, Berlin, 1971. 33

Friedman [8] Robert Friedman. Global smoothings of varieties with normal crossings. *Ann. of Math. (2)*, 118(1):75–114, 1983. 4, 20

LMB [9] Laurent Moret-Bailly Gérard Laumon. *Champs algébriques*, volume 39. Springer-Verlag, 2000. 24

Gri [10] Phillip Griffiths. Logarithmic hodge structures: Report on the work of kato-usui. Notes from a talk given at MSRI on March 18, 2009. `http://math.brown.edu/~abrmovic/Griffiths-log.pdf`. 31

EGA_IV [11] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I. *Inst. Hautes Études Sci. Publ. Math.*, (20):259, 1964. 8

hs [12] Mark Haiman and Bernd Sturmfels. Multigraded Hilbert schemes. *J. Algebraic Geom.*, 13(4):725–769, 2004. 37

Harris-Mumford [13] Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982. With an appendix by William Fulton. 46

Ionel-Parker1 [14] Eleny-Nicoleta Ionel and Thomas H. Parker. Relative Gromov-Witten invariants. *Ann. of Math. (2)*, 157(1):45–96, 2003. 45

Ionel-Parker2 [15] Eleny-Nicoleta Ionel and Thomas H. Parker. The symplectic sum formula for Gromov-Witten invariants. *Ann. of Math. (2)*, 159(3):935–1025, 2004. 45

FKato-def [16] Fumiharu Kato. Log smooth deformation theory. *Tohoku Math. J. (2)*, 48(3):317–354, 1996. 21

FKato [17] Fumiharu Kato. Log smooth deformation and moduli of log smooth curves. *Internat. J. Math.*, 11(2):215–232, 2000. 13, 14, 15, 16, 36, 43, 48

KKato [18] Kazuya Kato. Logarithmic structures of Fontaine-Illusie. In *Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988)*, pages 191–224. Johns Hopkins Univ. Press, Baltimore, MD, 1989. 5, 8, 11, 12, 14, 15, 17, 22

ToricSing [19] Kazuya Kato. Toric singularities. *Amer. J. Math.*, 116(5):1073–1099, 1994. 11

Kato-Usui-book [20] Kazuya Kato and Sampei Usui. *Classifying spaces of degenerating polarized Hodge structures*, volume 169 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. 32, 33, 35, 36

Kim [21] Bumsig Kim. Logarithmic stable maps. http://arxiv.org/abs/0807.3611, 2008. 47

llar-companion [22] János Kollár. Algebraic geometry. In Timothy Gowers, June Barrow-Green, and Imre Leader, editors, *The Princeton companion to mathematics*, pages 363–372. Princeton University Press, Princeton, NJ, 2008. 4

Kontsevich [23] Maxim Kontsevich. Enumeration of rational curves via torus actions. In *The moduli space of curves (Texel Island, 1994)*, volume 129 of *Progr. Math.*, pages 335–368. Birkhäuser Boston, Boston, MA, 1995. 45

Li-Ruan [24] An-Min Li and Yongbin Ruan. Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds. *Invent. Math.*, 145(1):151–218, 2001. 45, 46

Li [25] Jun Li. Stable morphisms to singular schemes and relative stable morphisms. *J. Differential Geom.*, 57(3):509–578, 2001. 45

i-degeneration [26] Jun Li. A degeneration formula of GW-invariants. *J. Differential Geom.*, 60(2):199–293, 2002. 45

Li-Tian [27] Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11(1):119–174, 1998. 45

Mil [28] J. S. Milne. Introduction to Shimura varieties. In *Harmonic analysis, the trace formula, and Shimura varieties*, volume 4 of *Clay Math. Proc.*, pages 265–378. Amer. Math. Soc., Providence, RI, 2005. 31

Mochizuki [29] Shinichi Mochizuki. The geometry of the compactification of the Hurwitz scheme. *Publ. Res. Inst. Math. Sci.*, 31(3):355–441, 1995. 47

Ogus [30] Arthur Ogus. Lectures on logarithmic algebraic geometry. TeXed notes, 2001. `http://math.berkeley.edu/~ogus/preprints/log_book/logbook.pdf`. 5, 6, 7, 10, 12

toricHilb [31] Martin Olsson. Logarithmic interpretation of the main component in toric Hilbert schemes. In *Curves and abelian varieties*, volume 465 of *Contemp. Math.*, pages 231–252. Amer. Math. Soc., Providence, RI, 2008. 37, 38, 39

LogStack [32] Martin C. Olsson. Logarithmic geometry and algebraic stacks. *Ann. Sci. École Norm. Sup. (4)*, 36(5):747–791, 2003. 22, 24, 25, 26, 27

Olsson2003 [33] Martin C. Olsson. Universal log structures on semi-stable varieties. *Tohoku Math. J. (2)*, 55(3):397–438, 2003. 19, 21, 22

K3 [34] Martin C. Olsson. Semistable degenerations and period spaces for polarized $K3$ surfaces. *Duke Math. J.*, 125(1):121–203, 2004. 37

LogCot [35] Martin C. Olsson. The logarithmic cotangent complex. *Math. Ann.*, 333(4):859–931, 2005. 13, 27

defStacks [36] Martin C. Olsson. Deformation theory of representable morphisms of algebraic stacks. *Mathematische Zeitschrift*, 253(1), 2006. 27, 30

Olsson-curves [37] Martin C. Olsson. (Log) twisted curves. *Compos. Math.*, 143(2):476–494, 2007. 40

cancmpt [38] Martin C. Olsson. *Compactifying moduli spaces for abelian varieties*, volume 1958 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2008. 37

MR705035 [39] Ulf Persson and Henry Pinkham. Some examples of nonsmoothable varieties with normal crossings. *Duke Math. J.*, 50(2):477–486, 1983. 20

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