Logarithmic geometry and rational curves

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Rational connectedness (Kollár-Miyaoka-Mori, Campana)

"A variety X is *rationally connected* if through any two points $x, y \in X$ there is a map $f : \mathbb{P}^1 \to X$ having x, y in its image."
Better: a variety X is rationally connected if there is a variety Y, and a morphism $F : Y \times \mathbb{P}^1 \to X$ such that the morphism

$$\begin{array}{ccc} Y & \to & X \times X \\ y & \mapsto & (F(y \times 0), F(y \times \infty)) \end{array}$$

is dominant.
For smooth projective varieties this is a birational property.

Very free curves

Assume X is smooth.
In characteristic 0, Sard's Theorem shows that for general $y \in Y$, the bundle $E_{F_y} := F_y^* T_X$ on $\mathbb{P}^1$ is globally generated. We say that X has *free rational curve*.
One can then show

Theorem (Kollár)

a smooth projective X in characteristic 0 is rationally connected if and only if there is a map $f : PP^1 \to X$ such that E_f is ample.

We say that X has a very free curve.

Separable rational connectedness

Assume X is smooth projective in positive characteristic. We say that X is *separably rationally connected* if there is F such that $Y \to X \times X$ is *dominant and generically separable*. (generically smooth suffices)

Theorem (Kollár)

a smooth projective X in characteristic p is separably rationally connected if and only if X has a very free curve.

This property is very powerful. It implies a whole lot on the geometry of X. It also implies general structure results, such as the boundedness of Fano varieties.

Examples

Clearly $\mathbb{P}^n$ is rationally connected. There is a chain of inclusions Rational $\subset$ stably rational $\subset$ unirational $\subset$ (separably) rationally connected.
All are known to be strict except the last!
A complex projective X is *Fano* if $-K_X$ is ample.

Theorem (KMM/C 92)

A Fano variety in characteristic 0 is rationally connected.

Question (Kollar)

Is a Fano variety in characteristic p separably rationally connected?

Examples

Theorem (Zhu 11)

A general Fano hypersurface is separably rationally connected.

Theorem (Chen-Zhu 13, Tian 13)

A general Fano complete intersection is separably rationally connected.

My goal is to present the approach of Chen-Zhu, relying on logarithmic geometry and degeneration. Tian's approach is very different (and elegant), using stability of tangent bundles..

Arithmetic relevance

Theorem (Graber-Harris-Starr, de Jong-Starr)

If X separably rationally connected over the function field K of a curve over an algebraically closed field, then $X(K) \neq \emptyset$.

Moduli of curves

$\mathcal{M}_g, \mathcal{M}_{0,n}$ - a quasiprojective variety.

> *Working with a non-complete moduli space is like keeping change in a pocket with holes*
>
> Angelo Vistoli

Deligne–Mumford

- $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ - moduli of *stable* curves, a modular compactification.
- allow only nodes as singularities
- What's so great about nodes?
- One answer: from the point of view of logarithmic geometry, these are the *logarithmically smooth* curves.

Logarithmic structures

Definition

A *pre logarithmic structure* is

$$X = (\underline{X}, M \stackrel{\alpha}{\to} \mathcal{O}_{\underline{X}}) \qquad \text{or just} \quad (\underline{X}, M)$$

such that

- $\underline{X}$ is a scheme - the *underlying scheme*
- M is a sheaf of monoids on X, and
- α is a monoid homomorphism, where the monoid structure on $\mathcal{O}_{\underline{X}}$ is the multiplicative structure.

Definition

It is a *logarithmic structure* if $\alpha : \alpha^{-1}\mathcal{O}_{\underline{X}}^* \to \mathcal{O}_{\underline{X}}^*$ is an isomorphism.

“Trivial” examples

Examples

- $(\underline{X}, \mathcal{O}^*_{\underline{X}} \hookrightarrow \mathcal{O}_{\underline{X}})$, the trivial *logarithmic* structure.
  We sometimes write just $\underline{X}$ for this structure.
- $(\underline{X}, \mathcal{O}_{\underline{X}} \xrightarrow{\sim} \mathcal{O}_{\underline{X}})$, looks as easy but surprisingly not interesting, and
- $(\underline{X}, \mathbb{N} \xrightarrow{\alpha} \mathcal{O}_{\underline{X}})$, where α is determined by an arbitrary choice of $\alpha(1)$.
  This one is important but only pre-logarithmic.

The associated logarithmic structure

You can always fix a pre-logarithmic structure:

$$\begin{tikzcd}
\alpha^{-1}\mathcal{O}^* \arrow[r, hook] \arrow[d] & M \arrow[d, dotted] \arrow[ddr, bend left, "\alpha"] & \\
\mathcal{O}^* \arrow[r, hook, dotted] \arrow[drr, bend right] & M^a \arrow[dr, "\alpha^a"] & \\
& & \mathcal{O}
\end{tikzcd}$$

[1]

[1] work out example $\mathbb{N}$ (next pages)

Key examples

Example (Divisorial logarithmic structure)

Let $\underline{X}, D \subset \underline{X}$ be a variety with a divisor. We define $M_D \hookrightarrow \mathcal{O}_{\underline{X}}$:

$$M_D(U) = \left\{ f \in \mathcal{O}_{\underline{X}}(U) \;\middle|\; f_{U \smallsetminus D} \in \mathcal{O}_{\underline{X}}^{\times}(U \smallsetminus D) \right\}.$$

This is particularly important for normal crossings divisors and toric divisors - these will be logarithmically smooth structures (a few pages down).

Example (Standard logarithmic point)

Let k be a field,

$$\begin{array}{rcl} \mathbb{N} \oplus k^\times & \to & k \\ (n, z) & \mapsto & z \cdot 0^n \end{array}$$

defined by sending $0 \mapsto 1$ and $n \mapsto 0$ otherwise.

Works with P a monoid with $P^\times = 0$, giving the *P-logarithmic point*. This is what you get when you restrict the structure on an affine toric variety associated to P to the maximal ideal generated by $\{p \neq 0\}$.

Morphisms

A morphism of (pre)-logarithmic schemes $f : X \to Y$ consists of

- $\underline{f} : \underline{X} \to \underline{Y}$
- A homomorphism $f^\flat$ making the following diagram commutative:

$$\begin{tikzcd} M_X \arrow[d, "\alpha_X"'] & \underline{f}^{-1}M_Y \arrow[l, "f^\flat"'] \arrow[d, "\alpha_Y"] \\ \mathcal{O}_{\underline{X}} & \underline{f}^{-1}\mathcal{O}_{\underline{Y}} \arrow[l, "\underline{f}^\sharp"'] \end{tikzcd}$$

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[2] automorphisms of a pointed point

Definition (**Inverse image**)

Given $\underline{f} : \underline{X} \to \underline{Y}$ and $Y = (\underline{Y}, M_Y)$ define the *pre-logarithmic inverse image* by composing

$$\underline{f}^{-1} M_Y \to \underline{f}^{-1}\mathcal{O}_{\underline{Y}} \xrightarrow{\underline{f}^\sharp} \mathcal{O}_{\underline{X}}$$

and then the *logarithmic inverse image* is defined as

$$\underline{f}^* M_Y = (\underline{f}^{-1} M_Y)^a.$$

This is the universal logarithmic structure on $\underline{X}$ with commutative

$$\begin{tikzcd} (\underline{X}, \underline{f}^* M_Y) \arrow[r] \arrow[d] & Y \arrow[d] \\ \underline{X} \arrow[r] & \underline{Y} \end{tikzcd}$$

$X \to Y$ is **strict** if $M_X = \underline{f}^* M_Y$.

Definition (Fibered products)

The fibered product $X \times_Z Y$ is defined as follows:

- $\underline{X \times_Z Y} = \underline{X} \times_{\underline{Z}} \underline{Y}$
- If N is the pushout of

$$\begin{tikzcd} & \pi_Z^{-1} M_Z \arrow[dl] \arrow[dr] & \\ \pi_X^{-1} M_X & & \pi_Y^{-1} M_Y \end{tikzcd}$$

then the log structure on $X \times_Z Y$ is defined by N^a.

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[3] remark on fp in fine, fine saturated categories

Definition (The spectrum of a Monoid algebras)

Let P be a monoid, R a ring. We obtain a monoid algebra $R[P]$ and a scheme $\underline{X} = \operatorname{Spec} R[P]$. There is an evident monoid homomorphism $P \to R[P]$ inducing sheaf homomorphism $P_X \to \mathcal{O}_{\underline{X}}$, a pre-logarithmic structure, giving rise to a logarithmic structure

$$(P_X)^a \to \mathcal{O}_{\underline{X}}.$$

This is a basic example. It deserves a notation:

$$X = \operatorname{Spec}(P \to R[P]).$$

The most basic example is $X_0 = \operatorname{Spec}(P \to \mathbb{Z}[P])$.

The morphism $\underline{f} : \operatorname{Spec}(R[P]) \to \operatorname{Spec}(\mathbb{Z}[P])$ gives

$$X = \underline{X} \times_{\underline{X_0}} X_0.$$

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[4] compare to toric divisors

Charts

A *chart* for X is given by a monoid P and a sheaf homomorphism $P_X \to \mathcal{O}_{\underline{X}}$ to which X is associated.
This is the same as a strict morphism $X \to \mathrm{Spec}(P \to \mathbb{Z}[P])$
Given a morphism of logarithmic schemes $f : X \to Y$, a chart for f is a triple

$$(P_X \to M_X, Q_Y \to M_Y, Q \to P)$$

such that

- $P_X \to M_X$ and $Q_Y \to M_Y$ are charts for M_X and M_Y, and
- the diagram

$$\begin{tikzcd} Q_X \arrow[r] \arrow[d] & f^{-1}M_Y \arrow[d] \\ P_X \arrow[r] & M_X \end{tikzcd}$$

  is commutative.

Types of logarithmic structures

- We say that $(\underline{X}, M_X)$ is *coherent* if *étale locally* at every point there is a *finitely generated* monoid P and a local chart $P_X \to \mathcal{O}_{\underline{X}}$ for X.[5]

- A monoid P is *integral* if $P \to P^{\mathrm{gp}}$ is injective.
- It is *saturated* if integral and whenever $p \in P^{\mathrm{gp}}$ and $m \cdot p \in P$ for some integrer $m > 0$ then $p \in P$. I.e., not like $\{0, 2, 3, \ldots\}$.[6]

- We say that a logarithmic structure is *fine* if it is *coherent* with local charts $P_X \to \mathcal{O}_X$ with P *integral*.
- We say that a logarithmic structure is *fine and saturated* (or fs) if it is coherent with local charts $P_X \to \mathcal{O}_X$ with P *integral and saturated*.

[5] example of $z = 0$ in $xy = zw$

[6] example of $\mathbb{A}^1$, cuspical curve

Definition (The characteristic sheaf)

Given a logarithmic structure $X = (\underline{X}, M)$, the quotient sheaf $\overline{M} := M/\mathcal{O}_{\underline{X}}^{\times}$ is called the *characteristic sheaf* of X.

The characteristic sheaf records the combinatorics of a logarithmic structure, especially for fs logarithmic structures.

Differentials

Say $T_0 = \operatorname{Spec} k$ and $T = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, and consider a morphism $X \to Y$.

We contemplate the following diagram:

$$
\begin{tikzcd}
& & X \arrow[d] \\
T_0 \arrow[r, hook] \arrow[urr, bend left, "\phi"] & T \arrow[ur, dotted, "g_1"] \arrow[ur, bend right, "g_0"'] \arrow[r, "\psi"'] & Y.
\end{tikzcd}
$$

Differentials (continued)

$$\begin{tikzcd}
 & & X \arrow[d] \\
T_0 \arrow[r, hook] \arrow[urr, bend left, "\phi"] & T \arrow[r, "\psi"'] \arrow[ur, bend right, "g_0"] \arrow[ur, dotted, "g_1"] & Y.
\end{tikzcd}$$

This translates to a diagram of groups and a diagram of monoids

$$\begin{tikzcd}
\mathbb{C} & \phi^{-1}\mathcal{O}_X \arrow[l] \arrow[dl, bend right, "g_0^\sharp"'] \arrow[dl, dotted, "g_1^\sharp"] \\
\mathbb{C}[\epsilon] \arrow[u] & \psi^{-1}\mathcal{O}_Y \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
M_{T_0} & \phi^{-1}M_X \arrow[l, "\phi^\flat"'] \arrow[dl, bend right, "g_0^\flat"'] \arrow[dl, dotted, "g_1^\flat"] \\
M_T \arrow[u] & \psi^{-1}M_Y \arrow[l] \arrow[u]
\end{tikzcd}$$

Differentials (continued)

$$
\begin{tikzcd}
\mathbb{C} & \phi^{-1}\mathcal{O}_X \arrow[l] \arrow[dl, "g_0^\sharp"'] \arrow[dl, dotted, bend left, "g_1^\sharp"] \\
\mathbb{C}[\epsilon] \arrow[u] & \psi^{-1}\mathcal{O}_Y \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
M_{T_0} & \phi^{-1}M_X \arrow[l, "\phi^\flat"'] \arrow[dl, "g_0^\flat"'] \arrow[dl, dotted, bend left, "g_1^\flat"] \\
M_T \arrow[u] & \psi^{-1}M_Y \arrow[l] \arrow[u]
\end{tikzcd}
$$

The difference $g_1^\sharp - g_0^\sharp$ is a derivation $\phi^{-1}\mathcal{O}_X \xrightarrow{d} \epsilon\mathbb{C} \simeq \mathbb{C}$
It comes from the sequence

$$0 \to J \to \mathcal{O}_{\underline{T}} \to \mathcal{O}_{\underline{T}_0} \to 0.$$

The multiplicative analogue

$$1 \to (1+J) \to \mathcal{O}_{\underline{T}}^\times \to \mathcal{O}_{\underline{T}_0}^\times \to 1$$

means, if all the logarithmic structures are integral,

$$1 \to (1+J) \to M_T \to M_{T_0} \to 1.$$

Differentials (continued)

$$
\begin{tikzcd}
\mathbb{C} & \phi^{-1}\mathcal{O}_X \arrow[l] \arrow[dl, "g_0^\sharp"'] \arrow[dl, dotted, bend left, "g_1^\sharp"] \\
\mathbb{C}[\epsilon] \arrow[u] & \psi^{-1}\mathcal{O}_Y \arrow[l] \arrow[u]
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
M_{T_0} & \phi^{-1}M_X \arrow[l, "\phi^\flat"'] \arrow[dl, "g_0^\flat"'] \arrow[dl, dotted, bend left, "g_1^\flat"] \\
M_T \arrow[u] & \psi^{-1}M_Y \arrow[l] \arrow[u]
\end{tikzcd}
$$

$$1 \to (1 + J) \to M_T \to M_{T_0} \to 1$$

means that we can take the "difference"

$$g_1^\flat(m) = (1 + D(m)) + g_0^\flat(m).$$

Namely $D(m) =$ "$g_1^\flat(m) - g_0^\flat(m)$" $\in J$.

Key properties:

- $D(m_1 + m_2) = D(m_1) + D(m_2)$
- $D|_{\psi^{-1}M_Y} = 0$
- $\alpha(m) \cdot D(m) = d(\alpha(m))$,

in other words,

$$D(m) \quad = \quad d \log\,(\alpha(m)),$$

which justifies the name of the theory.

Definition

A logarithmic derivation:

$$\begin{aligned} d : \mathcal{O} &\to J; \\ D : M &\to J \end{aligned}$$

satisfying the above.

Logarithmic derivations

Definition

A logarithmic derivation:

$$\begin{aligned} d : \mathcal{O} &\to J; \\ D : M &\to J \end{aligned}$$

satisfying the above.

The universal derivation:

$$d : \mathcal{O} \to \Omega^1_{\underline{X}/\underline{Y}} \quad = \quad \mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}/\text{relations}$$

The universal logarithmic derivation takes values in

$$\Omega^1_{X/Y} = \left(\Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O} \otimes_{\mathbb{Z}} M^{\mathrm{gp}})\right)/\text{relations}$$

Smoothness

Definition

We define a morphism $X \to Y$ of *fine logarithmic schemes* to be *logarithmically smooth* if

1. $\underline{X} \to \underline{Y}$ is locally of finite presentation, and
2. For T_0 fine and affine and $T_0 \subset T$ strict square-0 embedding, given

$$\begin{tikzcd} & & X \arrow[d] \\ T_0 \arrow[r, hook] \arrow[urr, bend left] & T \arrow[r] \arrow[ur, dotted, "\exists"] & Y \end{tikzcd}$$

   there exists a lifting as indicated.

The morphism is *logarithmically étale* if the lifting in (2) is unique.

Strict smooth morphisms

Lemma

If $X \to Y$ is strict and $\underline{X} \to \underline{Y}$ smooth then $X \to Y$ is logarithmically smooth.

Proof.

There is a lifting

$$\begin{tikzcd} & & \underline{X} \arrow[d] \\ \underline{T_0} \arrow[r, hook] \arrow[urr, bend left] & \underline{T} \arrow[r] \arrow[ur, dotted, "\exists"] & \underline{Y} \end{tikzcd}$$

since $\underline{X} \to \underline{Y}$ smooth, and the lifting of morphism of monoids comes by the universal property of pullback. ♠

Combinatirially smooth morphisms

Proposition

Say P, Q are finitely generated integral *monoids, R a ring, $Q \to P$ a monoid homomorphism.*
Write $X = \mathrm{Spec}(P \to R[P])$ and $Y = \mathrm{Spec}(Q \to R[Q])$.
Assume

- $\mathrm{Ker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ *is finite and with order invertible in R,*
- $\mathrm{TorCoker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ *has order invertible in R.*

Then $X \to Y$ is logarithmically smooth.
If also the cokernel is finite then $X \to Y$ is logarithmically étale.

(proof on board!)

Key examples

- Dominant toric morphisms
- Nodal curves.
- Marked nodal curves.
- $\operatorname{Spec}\mathbb{C}[t] \to \operatorname{Spec}\mathbb{C}[s]$ given by $s = t^2$
- $\operatorname{Spec}\mathbb{C}[x,y] \to \operatorname{Spec}\mathbb{C}[t]$ given by $t = x^m y^n$
- $\operatorname{Spec}\mathbb{C}[x,y] \to \operatorname{Spec}\mathbb{C}[x,z]$ given by $z = xy$.
- $\operatorname{Spec}(\mathbb{N} \to \mathbb{C}[\mathbb{N}]) \to \operatorname{Spec}((\mathbb{N}\smallsetminus 1) \to \mathbb{C}[(\mathbb{N}\smallsetminus 1)])$.

Integral morphisms

Disturbing feature: the last two examples are not flat. Which ones are flat? We define a monoid homomorphism $Q \to P$ to be *integral* if

$$\mathbb{Z}[Q] \to \mathbb{Z}[P]$$

is flat.

A morphism $f : X \to Y$ of logarithmic schemes is *integral* if for every geometric point x of X the homomorphism

$$(f^{-1}\overline{M}_Y)_x \to (\overline{M}_X)_x$$

of characteristic sheaves is integral.

Characterization of logarithmic smoothness

Theorem (K. Kato)

X, Y fine, $Q_Y \to M_Y$ a chart.
Then $X \to Y$ is logarithmically smooth iff there are extensions to local charts

$$(P_X \to M_X, Q_Y \to M_Y, Q \to P)$$

for $X \to Y$ such that

- *$Q \to P$ combinatorially smooth, and*
- *$\underline{X} \to \underline{Y} \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P]$ is smooth*

One direction:

$$\begin{tikzcd}
\underline{X} \arrow[r] & \underline{Y} \times_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[P] \arrow[r] \arrow[d] & \operatorname{Spec}\mathbb{Z}[P] \arrow[d] \\
& \underline{Y} \arrow[r] & \operatorname{Spec}\mathbb{Z}[Q]
\end{tikzcd}$$

Deformations

Proposition (K. Kato)

If $X_0 \to Y_0$ is logarithmically smooth, $Y_0 \subset Y$ a strict square-0 extension, then locally X_0 can be lifted to a smooth $X \to Y$.

Sketch of proof: locally $X_0 \to X_0' \to Y_0$, where

$$X_0' = Y_0 \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P].$$

So $X_0' \to Y_0$ is combinatorially smooth, and automatically provided a deformation to

$$Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P],$$

and $X_0 \to X_0'$ is strict and smooth so deforms by the classical result.

Kodaira-Spencer theory

Theorem (K. Kato)

Let Y_0 be artinian, $Y_0 \subset Y$ a strict square-0 extension with ideal J, and $f_0 : X_0 \to Y_0$ logarithmically smooth. Then

- *There is a canonical element $\omega \in H^2(X_0, T_{X_0/Y_0} \otimes f_0^* J)$ such that a logarithmically smooth deformation $X \to Y$ exists if and only if $\omega = 0$.*
- *If $\omega = 0$, then isomorphism classes of such $X \to Y$ correspond to elements of a torsor under $H^1(X_0, T_{X_0/Y_0} \otimes f_0^* J)$.*
- *Given such deformation $X \to Y$, its automorphism group is $H^0(X_0, T_{X_0/Y_0} \otimes f_0^* J)$.*

Corollary

Logarithmically smooth curves are unobstructed.

Saturated morphisms

Recall that the monoid homomorphism $\mathbb{N} \xrightarrow{\cdot 2} \mathbb{N}$ gives an integral logarithmically étale map with non-reduced fibers.

Definition

- An integral $Q \to P$ of saturated monoids is said to be *saturated* if $\mathrm{Spec}(P \to \mathbb{Z}[P]) \to \mathrm{Spec}(Q \to \mathbb{Z}[Q])$ has reduced fibers.
- An integral morphism $X \to Y$ of fs logarithmic schemes is *saturated* if it has a saturated chart.

This guarantees that if $X \to Y$ is logarithmically smooth, then the fibers are reduced.

Log curves

Definition

A *log curve* is a morphism $f : X \to S$ of fs logarithmic schemes satisfying:

- f is logarithmically smooth,
- f is integral, i.e. flat,
- f is saturated, i.e. has reduced fibers, and
- the fibers are curves i.e. pure dimension 1 schemes.

Theorem (F. Kato)

Assume $\pi : X \to S$ is a log curve. Then

- *Fibers have at most nodes as singularities*
- *étale locally on S we can choose disjoint sections $s_i : \underline{S} \to \underline{X}$ in the nonsingular locus $\underline{X}_0$ of $\underline{X}/\underline{S}$ such that*
    - *Away from s_i we have that $X^0 = \underline{X}_0 \times_{\underline{S}} S$, so π is strict away from s_i*
    - *Near each s_i we have a strict étale*

      $$X^0 \to S \times \mathbb{A}^1$$

      with the standard divisorial logarithmic structure on $\mathbb{A}^1$.
    - *étale locally at a node $xy = f$ the log curve X is the pullback of*

      $$\mathrm{Spec}(\mathbb{N}^2 \to \mathbb{Z}[\mathbb{N}^2]) \to Spec(\mathbb{N} \to \mathbb{Z}[\mathbb{N}])$$

      where $\mathbb{N} \to \mathbb{N}^2$ is the diagonal. Here the image of $1 \in \mathbb{N}$ in $\mathcal{O}_S$ is f and the generators of $\mathbb{N}^2$ map to x and y.

Log Fano complete intersections

We work in $\mathbb{P}^n$. Recall that $K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-(n+1))$.
Fix integers $(d_1, \ldots, d_l; d_b)$. Consider general hypersurfaces H_i of degrees $d_i, i = 1, \ldots l$ and their intersection $X = \cap H_i$.
Let H_b be a general hypersurface of degree d_b and consider $D = X \cap H_b$.
By the adjunction formula

$$K_X = \mathcal{O}_X(-(n+1)+\sum d_i) \quad \text{and} \quad K_X(D) = \mathcal{O}_X(-(n+1)+\sum d_i+d+b).$$

So X is Fano if $\sum d_i \leq n$ and (X, D) log Fano if $\sum d_i + d_b \leq n$.
It is consistent to allow $d_b = 0$.

Degeneration

Consider a degeneration of X, by allowing H_l to degenerate in a pencil into H_l' of degree $d_l - 1$ and H_l'' a hyperplane.
Get a family $\mathcal{X} \to \mathbb{A}^1$. The special fiber is $X_1 \cup_D X_2$. The family is log smooth away from $D \cap H_l$.
We can endow X_i with log structure coming from D:
(X_1, D) log Fano of type $(d_1, \ldots, d_{l-1}, d_l - 1; \ \ 1)$.
(X_2, D) log Fano of type $(d_1, \ldots, d_{l-1}, 1; \ \ d_l - 1)$.
D Fano of type $(d_1, \ldots, d_{l-1}, d_l - 1, 1)$.

Strategy

0 general D is SRC by induction on dimension
1 general (X_1, D) has a free line $(\mathbb{P}^1, 0)$ by an explicit construction (uses $1 = d_b$ prime to p).
2 One can lift a curve on D to (X_1, D) and glue with a curve in (X_1, D), and deform to a very free curve.
3 One can lift a free curve on D to a free $(\mathbb{P}^1, 0)$ on (X_2, D).
4 One can glue the two and deform to a very free curve on X.

1. Lines

Proposition

If $p \nmid d_b > 1$ there are free $(\mathbb{P}^1, 0)$ lines on X.

Idea of proof:

- Take one line L, say $x_2 = \cdots = x_n = 0$.
- Take "the general" equations of the form $F_i = y_0 x_1^{d_i - 1} + y_1 x_1^{d_i - 2} x_0 + \cdots y_{d_i - 2} x_1 x_0^{d_i - 2} + y_{d_i} x_0^{d_i - 1}$ and spread y_i among the x_i.
- These define X. for D take something like $G = X_1^{d_b} + y_1 x_1^{d_b - 2} x_0 + \cdots y_{d_b - 2} x_1 x_0^{d_b - 2} + y_{d_b} x_0^{d_b - 1}$
- Show regularity along L
- Show the normal bundle is non-negative.

2-3 Lifting curves

Proposition

Assume D SRC and $\mathcal{O}_X(D)$ ample. Then a free curve on D lifts to a free $(\mathbb{P}^1, 0)$ on (X, D).

idea of proof:

- works in special case of $\mathbb{P}(1 \oplus N_D)$.