

Logarithmic geometry and rational curves

Summer School 2015 of the IRTG "Moduli and Automorphic Forms"
Siena, Italy

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August 24-28, 2015

Rational connectedness (Kollár-Miyaoka-Mori, Campana)

“A variety X is *rationally connected* if through any two points $x, y \in X$ there is a map $f : \mathbb{P}^1 \rightarrow X$ having x, y in its image.”

Better: a variety X is rationally connected if there is a variety Y , and a morphism $F : Y \times \mathbb{P}^1 \rightarrow X$ such that the morphism

$$\begin{aligned} Y &\rightarrow X \times X \\ y &\mapsto (F(y \times 0), F(y \times \infty)) \end{aligned}$$

is dominant.

For smooth projective varieties this is a birational property.

Very free curves

Assume X is smooth.

In characteristic 0, Sard's Theorem shows that for general $y \in Y$, the bundle $E_{F_y} := F_y^* T_X$ on $\mathbb{P}^1$ is globally generated. We say that X has *free rational curve*.

One can then show

Theorem (Kollár)

a smooth projective X in characteristic 0 is rationally connected if and only if there is a map $f : \mathbb{P}^1 \rightarrow X$ such that E_f is ample.

We say that X has a very free curve.

Separable rational connectedness

Assume X is smooth projective in positive characteristic. We say that X is *separably rationally connected* if there is F such that $Y \rightarrow X \times X$ is dominant and generically separable. (generically smooth suffices)

Theorem (Kollár)

a smooth projective X in characteristic p is separably rationally connected if and only if X has a very free curve.

This property is very powerful. It implies a whole lot on the geometry of X . It also implies general structure results, such as the boundedness of Fano varieties.

Examples

Clearly $\mathbb{P}^n$ is rationally connected. There is a chain of inclusions
Rational $\subset$ stably rational $\subset$ unirational $\subset$ (separably) rationally connected.

All are known to be strict except the last!

A complex projective X is *Fano* if $-K_X$ is ample.

Theorem (KMM/C 92)

A Fano variety in characteristic 0 is rationally connected.

Question (Kollar)

Is a Fano variety in characteristic p separably rationally connected?

Examples

Theorem (Zhu 11)

A general Fano hypersurface is separably rationally connected.

Theorem (Chen-Zhu 13, Tian 13)

A general Fano complete intersection is separably rationally connected.

My goal is to present the approach of Chen-Zhu, relying on logarithmic geometry and degeneration. Tian's approach is very different (and elegant), using stability of tangent bundles..

Arithmetic relevance

Theorem (Graber-Harris-Starr, de Jong-Starr)

If X separably rationally connected over the function field K of a curve over an algebraically closed field, then $X(K) \neq \emptyset$.

Moduli of curves

$\mathcal{M}_g, \mathcal{M}_{0,n}$ - a quasiprojective variety.

*Working with a non-complete moduli space is like keeping
change in a pocket with holes*

Angelo Vistoli

Deligne–Mumford

- $\mathcal{M}_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ - moduli of *stable* curves, a modular compactification.
- allow only nodes as singularities
- What's so great about nodes?
- One answer: from the point of view of logarithmic geometry, these are the *logarithmically smooth* curves.

Logarithmic structures

Definition

A *pre logarithmic structure* is

$$X = (\underline{X}, M \xrightarrow{\alpha} \mathcal{O}_{\underline{X}}) \quad \text{or just} \quad (\underline{X}, M)$$

such that

- $\underline{X}$ is a scheme - the *underlying scheme*
- M is a sheaf of monoids on X , and
- α is a monoid homomorphism, where the monoid structure on $\mathcal{O}_{\underline{X}}$ is the multiplicative structure.

Definition

It is a *logarithmic structure* if $\alpha : \alpha^{-1}\mathcal{O}_{\underline{X}}^* \rightarrow \mathcal{O}_{\underline{X}}^*$ is an isomorphism.

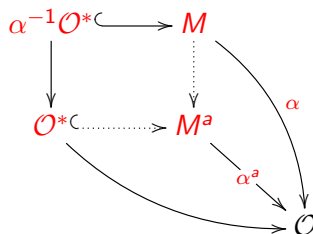
“Trivial” examples

Examples

- $(\underline{X}, \mathcal{O}_{\underline{X}}^* \hookrightarrow \mathcal{O}_{\underline{X}})$, the **trivial** *logarithmic* structure.
We sometimes write just $\underline{X}$ for this structure.
- $(\underline{X}, \mathcal{O}_{\underline{X}} \xrightarrow{\sim} \mathcal{O}_{\underline{X}})$, looks as easy but surprisingly not interesting, and
- $(\underline{X}, \mathbb{N} \xrightarrow{\alpha} \mathcal{O}_{\underline{X}})$, where α is determined by an arbitrary choice of $\alpha(1)$.
This one is important but only pre-logarithmic.

The associated logarithmic structure

You can always fix a pre-logarithmic structure:



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¹work out example $\mathbb{N}$ (next pages)

Key examples

Example (Divisorial logarithmic structure)

Let $\underline{X}, D \subset \underline{X}$ be a variety with a divisor. We define $M_D \hookrightarrow \mathcal{O}_{\underline{X}}$:

$$M_D(U) = \left\{ f \in \mathcal{O}_{\underline{X}}(U) \mid f_{U \setminus D} \in \mathcal{O}_{\underline{X}}^\times(U \setminus D) \right\}.$$

This is particularly important for normal crossings divisors and toric divisors
- these will be logarithmically smooth structures (a few pages down).

Example (Standard logarithmic point)

Let k be a field,

$$\begin{aligned}\mathbb{N} \oplus k^\times &\rightarrow k \\ (n, z) &\mapsto z \cdot 0^n\end{aligned}$$

defined by sending $0 \mapsto 1$ and $n \mapsto 0$ otherwise.

Works with P a monoid with $P^\times = 0$, giving the *P -logarithmic point*. This is what you get when you restrict the structure on an affine toric variety associated to P to the maximal ideal generated by $\{p \neq 0\}$.

Morphisms

A morphism of (pre)-logarithmic schemes $f : X \rightarrow Y$ consists of

- $\underline{f} : \underline{X} \rightarrow \underline{Y}$
- A homomorphism $f^\flat$ making the following diagram commutative:

$$\begin{array}{ccc} M_X & \xleftarrow{f^\flat} & \underline{f}^{-1} M_Y \\ \alpha_X \downarrow & & \downarrow \alpha_Y \\ \mathcal{O}_{\underline{X}} & \xleftarrow{\underline{f}^\sharp} & \underline{f}^{-1} \mathcal{O}_{\underline{Y}} \end{array}$$

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²automorphisms of a pointed point

Definition (Inverse image)

Given $\underline{f} : \underline{X} \rightarrow \underline{Y}$ and $\underline{Y} = (\underline{Y}, M_Y)$ define the *pre-logarithmic inverse image* by composing

$$\underline{f}^{-1} M_Y \rightarrow \underline{f}^{-1} \mathcal{O}_{\underline{Y}} \xrightarrow{\underline{f}^\sharp} \mathcal{O}_{\underline{X}}$$

and then the *logarithmic inverse image* is defined as

$$\underline{f}^* M_Y = (\underline{f}^{-1} M_Y)^a.$$

This is the universal logarithmic structure on $\underline{X}$ with commutative

$$\begin{array}{ccc} (\underline{X}, \underline{f}^* M_Y) & \longrightarrow & \underline{Y} \\ \downarrow & & \downarrow \\ \underline{X} & \longrightarrow & \underline{Y} \end{array}$$

$\underline{X} \rightarrow \underline{Y}$ is **strict** if $M_X = \underline{f}^* M_Y$.

Definition (Fibered products)

The fibered product $X \times_Z Y$ is defined as follows:

- $\underline{X \times_Z Y} = \underline{X} \times_{\underline{Z}} \underline{Y}$
- If N is the pushout of

$$\begin{array}{ccc} & \pi_Z^{-1} M_Z & \\ \swarrow & & \searrow \\ \pi_X^{-1} M_X & & \pi_Y^{-1} M_Y \end{array}$$

then the log structure on $X \times_Z Y$ is defined by N^a .

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³remark on fp in fine, fine saturated categories

Definition (The spectrum of a Monoid algebras)

Let P be a monoid, R a ring. We obtain a monoid algebra $R[P]$ and a scheme $\underline{X} = \operatorname{Spec} R[P]$. There is an evident monoid homomorphism $P \rightarrow R[P]$ inducing sheaf homomorphism $P_X \rightarrow \mathcal{O}_{\underline{X}}$, a pre-logarithmic structure, giving rise to a logarithmic structure

$$(P_X)^a \rightarrow \mathcal{O}_{\underline{X}}.$$

This is a basic example. It deserves a notation:

$$X = \operatorname{Spec}(P \rightarrow R[P]).$$

The most basic example is $X_0 = \operatorname{Spec}(P \rightarrow \mathbb{Z}[P])$.

The morphism $\underline{f} : \operatorname{Spec}(R[P]) \rightarrow \operatorname{Spec}(\mathbb{Z}[P])$ gives

$$X = \underline{X} \times_{\underline{X}_0} X_0.$$

Charts

A **chart** for X is given by a monoid P and a sheaf homomorphism $P_X \rightarrow \mathcal{O}_X$ to which X is associated.

This is the same as a strict morphism $X \rightarrow \mathrm{Spec}(P \rightarrow \mathbb{Z}[P])$

Given a morphism of logarithmic schemes $f : X \rightarrow Y$, a **chart for f** is a triple

$$(P_X \rightarrow M_X, Q_Y \rightarrow M_Y, Q \rightarrow P)$$

such that

- $P_X \rightarrow M_X$ and $Q_Y \rightarrow M_Y$ are charts for M_X and M_Y , and
- the diagram

$$\begin{array}{ccc} Q_X & \longrightarrow & f^{-1}M_Y \\ \downarrow & & \downarrow \\ P_X & \longrightarrow & M_X \end{array}$$

is commutative.

Types of logarithmic structures

- We say that $(\underline{X}, M_X)$ is *coherent* if *étale locally* at every point there is a *finitely generated* monoid P and a local chart $P_X \rightarrow \mathcal{O}_{\underline{X}}$ for X .⁵
- A monoid P is *integral* if $P \rightarrow P^{\text{gp}}$ is injective.
- It is *saturated* if integral and whenever $p \in P^{\text{gp}}$ and $m \cdot p \in P$ for some integer $m > 0$ then $p \in P$. I.e., not like $\{0, 2, 3, \dots\}$.⁶
- We say that a logarithmic structure is *fine* if it is *coherent* with local charts $P_X \rightarrow \mathcal{O}_X$ with P *integral*.
- We say that a logarithmic structure is *fine and saturated* (or *fs*) if it is coherent with local charts $P_X \rightarrow \mathcal{O}_X$ with P *integral and saturated*.

⁵example of $z = 0$ in $xy = zw$

⁶example of $\mathbb{A}^1$, cuspidal curve

Definition (The characteristic sheaf)

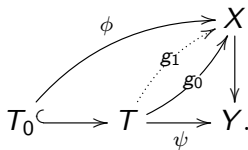
Given a logarithmic structure $X = (\underline{X}, M)$, the quotient sheaf $\overline{M} := M/\mathcal{O}_{\underline{X}}^\times$ is called the *characteristic sheaf* of X .

The characteristic sheaf records the combinatorics of a logarithmic structure, especially for fs logarithmic structures.

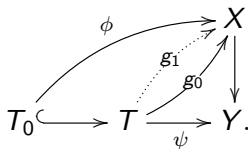
Differentials

Say $T_0 = \operatorname{Spec} k$ and $T = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, and consider a morphism $X \rightarrow Y$.

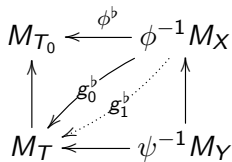
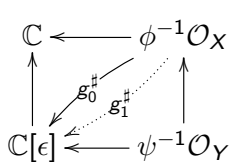
We contemplate the following diagram:



Differentials (continued)



This translates to a diagram of groups and a diagram of monoids



Differentials (continued)

$$\begin{array}{ccc}
 \mathbb{C} & \xleftarrow{\quad} & \phi^{-1}\mathcal{O}_X \\
 \uparrow & \swarrow g_0^\sharp & \uparrow \\
 \mathbb{C}[\epsilon] & \xleftarrow{\quad} & \psi^{-1}\mathcal{O}_Y
 \end{array}
 \qquad
 \begin{array}{ccc}
 M_{T_0} & \xleftarrow{\phi^b} & \phi^{-1}M_X \\
 \uparrow & \swarrow g_0^b & \uparrow \\
 M_T & \xleftarrow{\quad} & \psi^{-1}M_Y
 \end{array}$$

The difference $g_1^\sharp - g_0^\sharp$ is a **derivation** $\phi^{-1}\mathcal{O}_X \xrightarrow{d} \epsilon\mathbb{C} \simeq \mathbb{C}$
 It comes from the sequence

$$0 \rightarrow J \rightarrow \mathcal{O}_{\underline{T}} \rightarrow \mathcal{O}_{\underline{T}_0} \rightarrow 0.$$

The multiplicative analogue

$$1 \rightarrow (1 + J) \rightarrow \mathcal{O}_{\underline{T}}^\times \rightarrow \mathcal{O}_{\underline{T}_0}^\times \rightarrow 1$$

means, if all the logarithmic structures are integral,

$$1 \rightarrow (1 + J) \rightarrow M_T \rightarrow M_{T_0} \rightarrow 1.$$

Differentials (continued)

$$\begin{array}{ccc}
 \mathbb{C} & \xleftarrow{\quad} & \phi^{-1}\mathcal{O}_X \\
 \uparrow & \swarrow g_0^\# & \uparrow \\
 \mathbb{C}[\epsilon] & \xleftarrow{\quad} & \psi^{-1}\mathcal{O}_Y
 \end{array}$$

(Dotted arrow from $\mathbb{C}[\epsilon]$ to $\phi^{-1}\mathcal{O}_X$ labeled $g_1^\#$)

$$\begin{array}{ccc}
 M_{T_0} & \xleftarrow{\phi^b} & \phi^{-1}M_X \\
 \uparrow & \swarrow g_0^b & \uparrow \\
 M_T & \xleftarrow{\quad} & \psi^{-1}M_Y
 \end{array}$$

(Dotted arrow from M_T to $\phi^{-1}M_X$ labeled g_1^b)

$$1 \rightarrow (1 + J) \rightarrow M_T \rightarrow M_{T_0} \rightarrow 1$$

means that we can take the “difference”

$$g_1^b(m) = (1 + D(m)) + g_0^b(m).$$

Namely $D(m) = “g_1^b(m) - g_0^b(m)” \in J$.

Key properties:

- $D(m_1 + m_2) = D(m_1) + D(m_2)$
- $D|_{\psi^{-1}M_Y} = 0$
- $\alpha(m) \cdot D(m) = d(\alpha(m))$,

in other words,

$$D(m) = d \log (\alpha(m)),$$

which justifies the name of the theory.

Definition

A **logarithmic derivation**:

$$\begin{aligned} d : \mathcal{O} &\rightarrow J; \\ D : M &\rightarrow J \end{aligned}$$

satisfying the above.

Logarithmic derivations

Definition

A **logarithmic derivation**:

$$\begin{aligned} d : \mathcal{O} &\rightarrow J; \\ D : M &\rightarrow J \end{aligned}$$

satisfying the above.

The universal derivation:

$$d : \mathcal{O} \rightarrow \Omega_{\underline{X}/\underline{Y}}^1 = \mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O} / \text{relations}$$

The universal logarithmic derivation takes values in

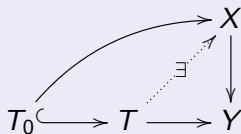
$$\Omega_{X/Y}^1 = \left(\Omega_{\underline{X}/\underline{Y}}^1 \oplus (\mathcal{O} \otimes_{\mathbb{Z}} M^{\text{gp}}) \right) / \text{relations}$$

Smoothness

Definition

We define a morphism $X \rightarrow Y$ of *fine logarithmic schemes* to be *logarithmically smooth* if

- 1 $\underline{X} \rightarrow \underline{Y}$ is locally of finite presentation, and
- 2 For T_0 fine and affine and $T_0 \subset T$ strict square-0 embedding, given



there exists a lifting as indicated.

The morphism is *logarithmically étale* if the lifting in (2) is unique.

Strict smooth morphisms

Lemma

If $X \rightarrow Y$ is *strict* and $\underline{X} \rightarrow \underline{Y}$ *smooth* then $X \rightarrow Y$ is logarithmically smooth.

Proof.

There is a lifting

A commutative diagram illustrating the lifting of a morphism. At the bottom, there is a sequence of monoids $\underline{T}_0 \hookrightarrow \underline{T} \rightarrow \underline{Y}$. Above $\underline{T}$ is a monoid $\underline{X}$, and above $\underline{Y}$ is a monoid X . A solid curved arrow goes from $\underline{T}_0$ to X . A solid straight arrow goes from $\underline{T}$ to X . A solid straight arrow goes from $\underline{X}$ to X . A solid straight arrow goes from $\underline{Y}$ to X . A dotted arrow labeled $\exists$ goes from $\underline{T}$ to $\underline{X}$.

since $\underline{X} \rightarrow \underline{Y}$ smooth, and the lifting of morphism of monoids comes by the universal property of pullback.



Combinatorially smooth morphisms

Proposition

Say P, Q are finitely generated **integral** monoids, R a ring, $Q \rightarrow P$ a monoid homomorphism.

Write $X = \operatorname{Spec}(P \rightarrow R[P])$ and $Y = \operatorname{Spec}(Q \rightarrow R[Q])$.

Assume

- $\operatorname{Ker}(Q^{\operatorname{gp}} \rightarrow P^{\operatorname{gp}})$ is finite and with order invertible in R ,
- $\operatorname{TorCoker}(Q^{\operatorname{gp}} \rightarrow P^{\operatorname{gp}})$ has order invertible in R .

Then $X \rightarrow Y$ is logarithmically smooth.

If also the cokernel is finite then $X \rightarrow Y$ is logarithmically étale.

(proof on board!)

Key examples

- Dominant toric morphisms
- Nodal curves.
- Marked nodal curves.
- $\mathrm{Spec} \mathbb{C}[t] \rightarrow \mathrm{Spec} \mathbb{C}[s]$ given by $s = t^2$
- $\mathrm{Spec} \mathbb{C}[x, y] \rightarrow \mathrm{Spec} \mathbb{C}[t]$ given by $t = x^m y^n$
- $\mathrm{Spec} \mathbb{C}[x, y] \rightarrow \mathrm{Spec} \mathbb{C}[x, z]$ given by $z = xy$.
- $\mathrm{Spec}(\mathbb{N} \rightarrow \mathbb{C}[\mathbb{N}]) \rightarrow \mathrm{Spec}((\mathbb{N} \setminus 1) \rightarrow \mathbb{C}[(\mathbb{N} \setminus 1)])$.

Integral morphisms

Disturbing feature: the last two examples are not flat. Which ones are flat?
We define a monoid homomorphism $Q \rightarrow P$ to be *integral* if

$$\mathbb{Z}[Q] \rightarrow \mathbb{Z}[P]$$

is flat.

A morphism $f : X \rightarrow Y$ of logarithmic schemes is *integral* if for every geometric point x of X the homomorphism

$$(f^{-1}\overline{M}_Y)_x \rightarrow (\overline{M}_X)_x$$

of characteristic sheaves is integral.

Characterization of logarithmic smoothness

Theorem (K. Kato)

X, Y *fine*, $Q_Y \rightarrow M_Y$ *a chart*.

Then $X \rightarrow Y$ is *logarithmically smooth* iff there are extensions to local charts

$$(P_X \rightarrow M_X, Q_Y \rightarrow M_Y, Q \rightarrow P)$$

for $X \rightarrow Y$ such that

- $Q \rightarrow P$ combinatorially smooth, and
- $\underline{X} \rightarrow \underline{Y} \times_{\mathrm{Spec} \mathbb{Z}[Q]} \mathrm{Spec} \mathbb{Z}[P]$ is smooth

One direction:

$$\begin{array}{ccccc} \underline{X} & \longrightarrow & \underline{Y} \times_{\mathrm{Spec} \mathbb{Z}[Q]} \mathrm{Spec} \mathbb{Z}[P] & \longrightarrow & \mathrm{Spec} \mathbb{Z}[P] \\ & & \downarrow & & \downarrow \\ & & \underline{Y} & \longrightarrow & \mathrm{Spec} \mathbb{Z}[Q] \end{array}$$

Deformations

Proposition (K. Kato)

If $X_0 \rightarrow Y_0$ is logarithmically smooth, $Y_0 \subset Y$ a strict square-0 extension, then locally X_0 can be lifted to a smooth $X \rightarrow Y$.

Sketch of proof: locally $X_0 \rightarrow X'_0 \rightarrow Y_0$, where

$$X'_0 = Y_0 \times_{\mathrm{Spec} \mathbb{Z}[Q]} \mathrm{Spec} \mathbb{Z}[P].$$

So $X'_0 \rightarrow Y_0$ is combinatorially smooth, and automatically provided a deformation to

$$Y \times_{\mathrm{Spec} \mathbb{Z}[Q]} \mathrm{Spec} \mathbb{Z}[P],$$

and $X_0 \rightarrow X'_0$ is strict and smooth so deforms by the classical result.

Kodaira-Spencer theory

Theorem (K. Kato)

Let Y_0 be artinian, $Y_0 \subset Y$ a strict square-0 extension with ideal J , and $f_0 : X_0 \rightarrow Y_0$ *logarithmically smooth*. Then

- There is a canonical element $\omega \in H^2(X_0, T_{X_0/Y_0} \otimes f_0^* J)$ such that a logarithmically smooth deformation $X \rightarrow Y$ exists if and only if $\omega = 0$.
- If $\omega = 0$, then isomorphism classes of such $X \rightarrow Y$ correspond to elements of a torsor under $H^1(X_0, T_{X_0/Y_0} \otimes f_0^* J)$.
- Given such deformation $X \rightarrow Y$, its automorphism group is $H^0(X_0, T_{X_0/Y_0} \otimes f_0^* J)$.

Corollary

Logarithmically smooth curves are unobstructed.

Saturated morphisms

Recall that the monoid homomorphism $\mathbb{N} \xrightarrow{\cdot 2} \mathbb{N}$ gives an integral logarithmically étale map with non-reduced fibers.

Definition

- An integral $Q \rightarrow P$ of saturated monoids is said to be *saturated* if $\mathrm{Spec}(P \rightarrow \mathbb{Z}[P]) \rightarrow \mathrm{Spec}(Q \rightarrow \mathbb{Z}[Q])$ has reduced fibers.
- An integral morphism $X \rightarrow Y$ of fs logarithmic schemes is *saturated* if it has a saturated chart.

This guarantees that if $X \rightarrow Y$ is logarithmically smooth, then the fibers are reduced.

Log curves

Definition

A **log curve** is a morphism $f : X \rightarrow S$ of fs logarithmic schemes satisfying:

- f is logarithmically smooth,
- f is integral, i.e. flat,
- f is saturated, i.e. has reduced fibers, and
- the fibers are curves i.e. pure dimension 1 schemes.

Theorem (F. Kato)

Assume $\pi : X \rightarrow S$ is a log curve. Then

- *Fibers have at most nodes as singularities*
- *étale locally on S we can choose disjoint sections $s_i : \underline{S} \rightarrow \underline{X}$ in the nonsingular locus $\underline{X}_0$ of $\underline{X}/\underline{S}$ such that*
 - ▶ *Away from s_i we have that $X^0 = \underline{X}_0 \times_{\underline{S}} S$, so π is strict away from s_i*
 - ▶ *Near each s_i we have a strict étale*

$$X^0 \rightarrow S \times \mathbb{A}^1$$

with the standard divisorial logarithmic structure on $\mathbb{A}^1$.

- ▶ *étale locally at a node $xy = f$ the log curve X is the pullback of*

$$\mathrm{Spec}(\mathbb{N}^2 \rightarrow \mathbb{Z}[\mathbb{N}^2]) \rightarrow \mathrm{Spec}(\mathbb{N} \rightarrow \mathbb{Z}[\mathbb{N}])$$

where $\mathbb{N} \rightarrow \mathbb{N}^2$ is the diagonal. Here the image of $1 \in \mathbb{N}$ in $\mathcal{O}_S$ is f and the generators of $\mathbb{N}^2$ map to x and y .

Log Fano complete intersections

We work in $\mathbb{P}^n$. Recall that $K_{\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(-(n+1))$.

Fix integers $(d_1, \dots, d_l; d_b)$. Consider general hypersurfaces H_i of degrees $d_i, i = 1, \dots, l$ and their intersection $X = \cap H_i$.

Let H_b be a general hypersurface of degree d_b and consider $D = X \cap H_b$.
By the adjunction formula

$$K_X = \mathcal{O}_X(-(n+1) + \sum d_i) \quad \text{and} \quad K_X(D) = \mathcal{O}_X(-(n+1) + \sum d_i + d_b).$$

So X is Fano if $\sum d_i \leq n$ and (X, D) log Fano if $\sum d_i + d_b \leq n$.

It is consistent to allow $d_b = 0$.

Degeneration

Consider a degeneration of X , by allowing H_I to degenerate in a pencil into H'_I of degree $d_I - 1$ and H''_I a hyperplane.

Get a family $\mathcal{X} \rightarrow \mathbb{A}^1$. The special fiber is $X_1 \cup_D X_2$. The family is log smooth away from $D \cap H_I$.

We can endow X_i with log structure coming from D :

(X_1, D) log Fano of type $(d_1, \dots, d_{I-1}, d_I - 1; -1)$.

(X_2, D) log Fano of type $(d_1, \dots, d_{I-1}, 1; d_I - 1)$.

D Fano of type $(d_1, \dots, d_{I-1}, d_I - 1, 1)$.

Strategy

- 0 general D is SRC by induction on dimension
- 1 general (X_1, D) has a free line $(\mathbb{P}^1, 0)$ by an explicit construction (uses $1 = d_b$ prime to p).
- 2 One can lift a curve on D to (X_1, D) and glue with a curve in (X_1, D) , and deform to a very free curve.
- 3 One can lift a free curve on D to a free $(\mathbb{P}^1, 0)$ on (X_2, D) .
- 4 One can glue the two and deform to a very free curve on X .

1. Lines

Proposition

If $p \nmid d_b > 1$ there are free $(\mathbb{P}^1, 0)$ lines on X .

Idea of proof:

- Take **one** line L , say $x_2 = \cdots = x_n = 0$.
- Take "the general" equations of the form
$$F_i = y_0 x_1^{d_i-1} + y_1 x_1^{d_i-2} x_0 + \cdots y_{d_i-2} x_1 x_0^{d_i-2} + y_{d_i} x_0^{d_i-1}$$
and spread y_i among the x_i .
- These define X . for D take something like
$$G = X_1^{d_b} + y_1 x_1^{d_b-2} x_0 + \cdots y_{d_b-2} x_1 x_0^{d_b-2} + y_{d_b} x_0^{d_b-1}$$
- Show regularity along L
- Show the normal bundle is non-negative.

2-3 Lifting curves

Proposition

Assume D SRC and $\mathcal{O}_X(D)$ ample. Then a free curve on D lifts to a free $(\mathbb{P}^1, 0)$ on (X, D) .

idea of proof:

- works in special case of $\mathbb{P}(1 \oplus N_D)$.