LOGARITHMIC MAPS TO DELIGNE-FALTINGS PAIRS

DAN ABRAMOVICH AND QILE CHEN

Contents

1. Introduction

Compared with the previous version, the following changes are made:

(1) The proof for boundedness is rewritten, instead of gluing the node, we analyze the corresponding line bundle. (See subsection 7.3).
(2) The proof for boundedness and valuative criterion is now reduce to the one copy of $\mathbb{N}$ case.
(3) A short discussion for general DF-log structure can be find in subsection 2.2.

2. Prerequisites on logarithmic geometry

2.1. **Basic definitions and properties.** Following [Kat89] and [Ogu06], we first recall some basic terminologies on logarithmic geometry.

2.1.1. *Monoids.* A *monoid* is a commutative semi-group with a unit. We usually use " + " and "0" denote the binary operation and the unit of a monoid. A *morphism between two monoids* is required to preserve the unit.

Let P be a monoid, we can associate a group

$$P^{gp} := \{(a,b) | (a,b) \sim (c,d) \text{ if } \exists s \in P \text{ such that } s+a+d = s+b+c\}.$$

We recall some terminologies:

(1) P is called *integral* if the natural map $P \to P^{gp}$ is injective.
(2) P is called *saturated* if it is integral and satisfies that for any $p \in P^{gp}$, if $n \cdot p \in P$ for some positive integer n then $p \in P$.
(3) P is *fine* if it is integral and finitely generated.

Date: May 5, 2010.

(4) P is *sharp* if there are no other unit except 0. A nonzero element p in a sharp monoid P is called *irreducible* if $p = a + b$ implies either $a = 0$ or $b = 0$. We denote by $Irr(P)$ the set of irreducible elements in a sharp monoid P.
(5) A fine monoid P is called *free* if $P \cong \mathbb{N}^n$ for some positive integer n.
(6) A monoid P is called torsion free if the associated group P^{gp} is torsion free.
(7) The monoid P is called toric if P is fine, saturated, and sharp. Note that in this case p is automatically torsion free.

Denote by Mon^{int} and Mon^{sat} the categories of integral and saturated monoids respectively. Then there is an natural inclusion

$$\iota : Mon^{sat} \to Mon^{int}.$$

On the other hand, given a integral monoid M, the set M^{sat} of all elements $a \in M^{gp}$ such that $m \cdot a \in M$ for some positive integer m forms a saturated submonoid of M^{gp}. This induces another map

$$\mathcal{S}at : Mon^{int} \to Mon^{sat}.$$

rop:AdjSat **Proposition 2.1.** [Ogu06, 1.2.3(3)] *The functor $\mathcal{S}at$ is left adjoint to the functor ι.*

A morphism $h : Q \to P$ between integral monoids is called *integral* if for any $a_1, a_2 \in Q$, and $b_1, b_2 \in P$ which satisfy $h(a_1)b_1 = h(a_2)b_2$, there exist $a_2, a_4 \in Q$ and $b \in P$ such that $b_1 = h(a_3)b$ and $a_1a_3 = a_2a_4$.

2.1.2. *Congruence relation and finite representation of monoids.* Consider a morphism of monoids $q : P \to Q$. We form the following set

uenceOfMap (2.1.1) $$E := \{\ (p_1, p_2) \in P \times P \mid q(p_1) = q(p_2)\ \} \subset P \times P.$$

It is not hard to check that the set E is a submonoid of $P \times P$, which gives an equivalence relation on P. If q is surjective, then the monoid Q can be recovered as the quotient of P by the equivalence relation E. In this case, we write $Q \cong P/E$. A submonoid $E \subset P \times P$ is called a *congruence relation on* P, if it is a equivalence relation on P. Conversely, given a congruence relation E on P, we have a canonical surjective morphism of monoids $q : P \to P/E$, such that E is of the form as in (2.1.1).

A *presentation of a monoid* M is a diagram

MonPresent (2.1.2) $$F_1 \underset{v}{\overset{u}{\rightrightarrows}} F_0 \xrightarrow{q} M,$$

where F_0 and F_1 are free, and q is the coequalizer of u and v. If furthermore F_0 and F_1 is finitely generated, then (2.1.2) is called a finite presentation of M. Given a monoid M with the presentation as in (2.1.2), we can recover M as the quotient of F_0 given by the congruence relation

$$E := \{\ (u(a), v(a)) \in F_0 \times F_0 \mid a \in F_1\ \}.$$

TorPresent **Remark 2.2.** Consider a toric monoid P. Denote by $Irr(P) = \{\delta_i\}_{i=1}^k$ the set of irreducible elements in P. Consider the free monoid $M_0 \cong N^k$ with the map of monoids

$$q : M_0 \to P, \quad \delta_{0,i} \mapsto \delta_i,$$

where $\{\delta_{0,i}\}_{i=1}^{k}$ forms a basis of $\mathbb{N}^k$. Since $Irr(P)$ generates P, the map q is surjective. By [Ogu06, Chapter 1, 2.1.9(7)], we have a finite presentation

TorPresent
$$M_1 \underset{v_2}{\overset{v_1}{\rightrightarrows}} M_0 \xrightarrow{\ q\ } P. \tag{2.1.3}$$

Since P is sharp, if $v_1(e) = 0$ for some $e \in M_1$, then we can check that $v_2(e) = 0$. We call diagram (2.1.3) constructed above the *standard presentation of* P, if $v_1(e)$ and $v_2(e)$ is non-trivial for any $0 \neq e \in M_1$. Denote by $\{\delta_{1,j}\}_{j=1}^{r}$ the set of basis of M_1, then we can write

onOfMonoid
$$P := \Big\langle\ \delta_1, \cdots, \delta_k \mid \gamma_j : q \circ u(\delta_{1,j}) = q \circ v(\delta_{1,j}),\ j = 1, \cdots, r\Big\rangle, \tag{2.1.4}$$

where γ_j stands for the corresponding relation.

:DefLogStr

2.1.3. *Logarithmic structures.* Let X be a scheme. A *pre-log structure* on X is a pair $(\mathcal{M}, \exp)$, which consists of a sheaf of monoids $\mathcal{M}$ on the étale site $X_{ét}$ of X, and a morphism of sheaves of monoids $\exp : \mathcal{M} \to \mathcal{O}_X$, called the structure morphism of $\mathcal{M}$. Here we view $\mathcal{O}_X$ as a monoid under multiplication.

A pre-log structure $\mathcal{M}$ on X is called a *log structure* if $\exp^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$ via exp. We sometimes omite the morphism exp, and only use $\mathcal{M}$ to denote the log structure if no confusion could arise. We call the pair $(X, \mathcal{M})$ a *log scheme*.

Given two log structures $\mathcal{M}$ and $\mathcal{N}$ on X, a *morphism of the log structures* $h : \mathcal{M} \to \mathcal{N}$ is a morphism of sheaves of monoids which compatible with the structure morphisms of $\mathcal{M}$ and $\mathcal{N}$.

Given a pre-log strucutre $\mathcal{M}$ on X, we can associate a log structure $\mathcal{M}^a$ given by

$$\mathcal{M}^a := \mathcal{M} \oplus_{\exp^{-1}(\mathcal{O}_X^*)} \mathcal{O}_X^*.$$

Consider a morphism of schemes $f : X \to Y$, and a log structure $\mathcal{M}_Y$ on Y. We can define the *pull-back log structure* $f^*(\mathcal{M}_Y)$ to be the log structure associated to the pre-log structure

$$f^{-1}(\mathcal{M}_Y) \to f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X.$$

Consider two log schemes $(X, \mathcal{M}_X)$ and $(Y, \mathcal{M}_Y)$. A morphism of log schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ is a pair $(f, f^\flat)$, where $f : X \to Y$ is a morphism of the underlying schemes, and $f^\flat : f^*(\mathcal{M}_Y) \to \mathcal{M}_X$ is a morphism of log structures on X. The morphism $(f, f^\flat)$ is called *strict* if $f^\flat$ is an isomorphism of log structures. It is called *vertical* if $\mathcal{M}_X/f^*(\mathcal{M}_Y)$ is a sheaf of groups under the induced monoidal operation.

s:ChartLog

2.1.4. *Charts of log structures.* Let $(X, \mathcal{M})$ be a log scheme, and P a monoid. Denote by P_X the constant sheaf of monoid P on X. A chart of $\mathcal{M}$ is a morphism $P_X \to \mathcal{M}$ such that the associated log structure of the composition $P_X \to \mathcal{M} \to \mathcal{O}_X$ is $\mathcal{M}$. The log structure $\mathcal{M}$ is called a *fine (resp. coherent) log structure* on X if P is fine (resp. coherent). If the monoid P is fs, then $\mathcal{M}$ is called a *fs log structure*. In this and the following sections, we will only consider fine log structures.

m:ChartMap

Remark 2.3. For any fs monoid Q, denote by $\mathrm{Spec}(Q \to \mathbb{Z}[Q])$ the log scheme with underlying $\mathrm{Spec}\mathbb{Z}[Q]$, and log structure induced by $Q \to \mathbb{Z}[Q]$. Any log structure $\mathcal{M}$ on X with chart $Q \to \mathcal{M}$ is equivalent to have a map $X \to \mathrm{Spec}\mathbb{Z}[Q]$ with $\mathcal{M}$ obtained by the pull-back of the log structure of $\mathrm{Spec}(Q \to \mathbb{Z}[Q])$.

Let $\overline{\mathcal{M}} = \mathcal{M}/\mathcal{O}_X^*$ be the quotient sheaf. We call it the characteristic of the log structure $\mathcal{M}$. It is useful to notice that $\overline{f^*(\mathcal{M})} = f^{-1}(\overline{\mathcal{M}})$ for any morphism of schemes $f : Y \to X$. For any closed point $x \in X$, we denote by $\bar{x}$ the separable closure of x. A fine log structure $\mathcal{M}$ is called locally free if for any $x \in X$, we have $\overline{\mathcal{M}}_{\bar{x}} \cong \mathbb{N}^n$ for some positive integer r. Let $\overline{\mathcal{M}}_{\bar{x}}^{gp,tor}$ be the torsion part of $\overline{\mathcal{M}}_{\bar{x}}^{gp}$. The following result is very useful for creating charts.

hartLogStr **Proposition 2.4.** [Ols03a, 2.1] *Using the notation as above, there exist an fppf neighborhood $f : X' \to X$ of x, and a chart $\beta : P \to f^*(\mathcal{M})$ such that for some geometric point $\bar{x}' \to X'$ lying over x, the natural map $P \to f^{-1}\overline{\mathcal{M}}_{\bar{x}'}$ is bijective. If $\overline{\mathcal{M}}_{\bar{x}}^{gp,tor} \otimes k(x) = 0$, then such a chart exists in an étale neighborhood of x.*

Remark 2.5. In the following sections, we will mostly work with fs log structures over an algebraicly closed field of characteristic 0. The above proposition implies that in such situation, there is a section of $\mathcal{M}_{\bar{x}} \to \overline{\mathcal{M}}_{\bar{x}}$, which can be lift to a chart étale locally near x.

Consider a morphism $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$ of fine log schemes. A chart of f is a triple $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ where $P_X \to \mathcal{M}_X$ and $Q_Y \to \mathcal{M}_Y$ are charts of $\mathcal{M}_X$ and $\mathcal{M}_Y$ respectively, and $Q \to P$ is a morphism of monoids such that the following diagram is commutative:

$$\begin{tikzcd} Q_X \arrow[r] \arrow[d] & P_X \arrow[d] \\ f^*(\mathcal{M}_Y) \arrow[r] & \mathcal{M}_X. \end{tikzcd}$$

Similarly, the charts of morphism of fine log schemes exist étale locally by the following result:

Proposition 2.6. [Ols03a, 2.2] *Notations as above, suppose that $Q_Y \to \mathcal{M}_Y$ is a chart. Then étale locally on X, there exist a chart $P_X \to \mathcal{M}_X$ and an injective morphism of monoids $Q \to P$, such that the triple $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ gives a chart for f étale locally on X. If f is a morphism of fs log schemes and if Q is saturated and torsion free, then we can choose P to be also saturated and torsion free in the chart of f.*

m:LogSmCri **Remark 2.7.** Consider a morphism of log schemes $f : (X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$, with the help of charts, we can describe the log smoothness properties of f that we will use later. The log map f is called *log smooth* if étale locally, there is a chart $(P_X \to \mathcal{M}_X, Q_Y \to \mathcal{M}_Y, Q \to P)$ of f such that:

1. $Ker Q^{gp} \to P^{gp}$ and the torsion part of $Coker(Q^{gp} \to P^{gp})$ are finite groups;
2. the induced map $X \to Y \times_{\mathrm{Spec}(\mathbb{Z}[Q])} \mathrm{Spec}\mathbb{Z}[p]$ is smooth in the usual sense.

The above smoothness criterion is due to K. Kato [Kat89, Theorem 3.5].

The map f is called *integral* if for every $p \in X$, the induced map $\overline{\mathcal{M}}_{f(\bar{p})} \to \overline{\mathcal{M}}_{\bar{p}}$ is integral. In general, the underlying structure map of a log smooth morphism need not be flat. However, it is shown in [Kat89, 4.5] that the underlying map of a log smooth and integral morphism is flat. Finally, we introduce an important result we will use later.

:SatFinite **Proposition 2.8.** [Ogu06, Chapter 2, 2.4.5]

1. *The inclusion functor from the category of fine log schemes to the category of coherent log schemes admits a right adjoint $X \mapsto X^{int}$, where X is a coherent log schemes.*

Furthermore, the corresponding morphism of underlying schemes $\underline{X}^{int} \to \underline{X}$ *is a closed immersion. We call* X^{int} *the integration of* X*.*

(2) *The inclusion functor from the category of fs log schemes to the category of fine log schemes admits a right adjoint* $X \mapsto X^{sat}$*, where* X *is a fine log scheme. Furthermore, the corresponding morphism of underlying schemes* $\underline{X}^{sat} \to \underline{X}$ *is finite and surjective. We call* X^{sat} *the saturation of* X*.*

ss:DFLog

2.2. Deligne-Faltings log structures.

defn:DF **Definition 2.9.** Consider a scheme X. A fs log structure $\mathcal{M}_X$ on X is called a *Deligne-Faltings (DF) log structures*, if there is a morphism of locally constant sheaves of monoids $\beta : P \to \overline{\mathcal{M}}_X$, which locally lifts to a chart. Here P is a toric monoid. We call the map β a *global presentation of* $\mathcal{M}_X$.

Remark 2.10. The global presentation β of a DF log structure $\mathcal{M}_X$ is not unique. But we will see later that our definition of minimality does not depend on the choice of β.

Remark 2.11. Notations as in definition 2.9, if an element $\delta \in Irr(P)$ satisfies $\beta(\delta) = 0$ everywhere, then we can choose a submonoid $P' \subset P$ generated by $Irr(P) \setminus \{\delta\}$, and we have a global presentation $\beta' : P' \to \mathcal{M}_X$ induced by β. Thus, we always require P to satisfy the condition that if $0 \neq \delta \in P$, then $\beta(\delta) \neq 0$.

DecomDFlog **Remark 2.12.** Denote by $\mathcal{M}_X^i$ the sub-log structure of $\mathcal{M}_X$ generated by δ_i. Then by definition $\mathcal{M}_X^i$ is a DF log structure on X, and we have

$$\mathcal{M}_X \cong \mathcal{M}_X^1 \oplus_{\mathcal{O}_X^*} \mathcal{M}_X^2 \oplus_{\mathcal{O}_X^*} \cdots \oplus_{\mathcal{O}_X^*} \mathcal{M}_X^k.$$

Denote by $X_i^{log} = (X, \mathcal{M}_X^i)$. Then the above decomposition is equivalent to the fiber product of fine log schemes:

FrDFTarget

$$(X, \mathcal{M}_X) \cong X_1^{log} \times_X \cdots \times_X X_k^{log}, \tag{2.2.1}$$

where X is viewed as a the log scheme with underlying X with trivial log structures.

neBundleDF **Remark 2.13.** Assume that the DF log structure $\mathcal{M}_X$ is locally free, then we can assume that $P \cong \mathbb{N}^k$. Denote by $\{\delta_i\}_{i=1}^k$ the standard generators of $\mathbb{N}^k$. Then locally we have a lifting $\tilde{\beta} : \mathbb{N}^k \to \mathcal{M}_X$. Note that the section $\beta(\delta_i)$ with its inverse image under the canonical map $\pi : \mathcal{M}_X \to \overline{\mathcal{M}}_X$ is a $\mathcal{O}_X^*$-torsor, which corresponds to a line bundle L_i. The composition

$$\pi^{-1}\beta(\delta_i) \subset \mathcal{M}_X \to \mathcal{O}_X$$

gives a morphism of line bundles $s_i : L_i \to \mathcal{O}_X$. In fact, it was shown in [Kat89, Complement 1] that a locally free DF log structure as above is equivalent to have k-tuple of line bundles $(L_i)_{i=1}^k$ with sections $s_i : L_i \to \mathcal{O}_X$ for each i.

Note that the section s_i gives a section s_i' of $L_i^\vee$. Denote by $D_i \subset X$ the vanishing locus of s_i'. Note that D_i consists of the points where the image of δ_i in $\overline{\mathcal{M}}_X$ is non-trivial. If s_i is not a zero section, then D_i is a Cartier divisor in X. If s_i is a zero section, then $D_i = X$, we call $\mathcal{M}_X^i$ the *generic part of* $\mathcal{M}_X$. Note that if $D_i = \emptyset$, then the sub-log structure generated by δ_i is trivial.

eg:SNC **Example 2.14.** Consider a simple normal crossing divisor $D \subset X$, then the following

$$\mathcal{M}_X = \{\ g \in \mathcal{O}_X \mid g \text{ is invertible outside } D\}$$

with the natural injection $\mathcal{M}_X \to X$ forms a DF log structure on X. Its rank k equals the number of irreducible components of D.

Underlying **Remark 2.15.** Consider a log smooth scheme $(X, \mathcal{M}_X)$, and assume that $\mathcal{M}_X$ is a locally free DF log structure on X. By the description of log smoothness in remark 2.7, the underlying scheme X is automatically smooth in the usual sense, and the log structure $\mathcal{M}_X$ is the one described in example 2.14. Note that in this case, $\mathcal{M}_X$ has no generic part.

Consider a DF log structure $\mathcal{M}_X$ and a global presentation $\beta : P \to \mathcal{M}_X$ as in definition 2.9. Consider an element $\delta \in P$. Since β locally lifts to a chart, the sub-monoid $\mathbb{N} \subset P$ generated by δ gives a rank one locally free sub-DF log structure $\mathcal{N}_i \subset \mathcal{M}_X$. Note that there is a global presentation $\mathbb{N} \to \overline{\mathcal{N}}_i$ induced by δ.

We use the notations as in remark 2.2. Denote by $\mathcal{N}_i$ the sub-log structure induced by $\delta_i \in Irr(P)$ as above. Consider the locally free DF log structures on X given by

$$\mathcal{M}_0 := \sum_{\delta_i \in Irr(P)} \mathcal{N}_i,$$

where the amalgamated sum is taking over $\mathcal{O}_X^*$. Note that we have a global presentation $\beta_0 : M_0 \cong \mathbb{N}^k \to \overline{\mathcal{M}}_0$, and a natural morphism $\tilde{q} : \mathcal{M}_0 \to \mathcal{M}_X$ induced by each $\mathcal{N}_i \to \mathcal{M}_X$. Now we repeat the same argument for the map of monoids $v_2 \circ q = v_1 \circ q$ as in (2.1.3), we have another locally free DF log structure $\mathcal{M}_1$, and a morphism of log structures $\phi : \mathcal{M}_1 \to \mathcal{M}_X$. A local calculation shows that we have the following diagram of log structures on X:

:DFPresent (2.2.2)
$$\mathcal{M}_1 \overset{v_1^\flat}{\underset{v_2^\flat}{\rightrightarrows}} \mathcal{M}_0 \xrightarrow{q^\flat} \mathcal{M}_X \ ,$$

such that $v_1^\flat \circ q^\flat = v_2^\flat \circ q^\flat = \phi$. Denote by $X_1^{log} = (X, \mathcal{M}_1), X_0^{log} = (X, \mathcal{M}_0)$, and $X^{log} = (X, \mathcal{M}_X)$. Note that $q^\flat$ is a surjection of sheaves of monoid. Then (2.2.2) induces a morphism of log schemes

SchPresent (2.2.3)
$$X^{log} \xrightarrow{q} X_0^{log} \overset{v_1}{\underset{v_2}{\rightrightarrows}} X_1^{log} \ ,$$

We call (2.2.3) constructed above the *locally free presentation of* X^{log}. Here we abuse the notations, and denote q, v_1 and v_2 the morphism of corresponding log schemes rather then the monoids as in (2.1.3).

compTarget **Lemma 2.16.** *We have a caterian diagram in the category of fs log schemes:*

$$\begin{tikzcd} X^{log} \arrow[r, "q'"] \arrow[d, "q'"'] & X_0^{log} \arrow[d, "v_2'"] \\ X_0^{log} \arrow[r, "v_1'"] & X_1^{log}. \end{tikzcd}$$

Proof. This is a local question, so we can assume that X is affine with global charts $M_0 \to \mathcal{M}_0$, $M_1 \to \mathcal{M}_1$, and $P \to \mathcal{M}_X$. Using remark 2.3, we have the following commutative diagram:

:MapDefLog (2.2.4)

$$\begin{tikzcd} X \arrow[dr, "g"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "f"'] & & \\ & \mathrm{Spec}\mathbb{Z}[P] \arrow[r] \arrow[d] & \mathrm{Spec}\mathbb{Z}[M_0] \arrow[d] \\ & \mathrm{Spec}\mathbb{Z}[M_0] \arrow[r] & \mathrm{Spec}\mathbb{Z}[M_1] \end{tikzcd},$$

where the square induced by the map of monoids in (2.1.3) is cartesian, and the arrows f and g is induced by the log structures $\mathcal{M}_0$ and $\mathcal{M}_X$ respectively. Note that the composition $X \to \mathbb{Z}[M_0] \to \mathbb{Z}[M_1]$ corresponds to the log structure $\mathcal{M}_1$, and the map g is induced by the map f and the universal property of fiber product. By (2.1.3) again, we have a cartesian diagram of fs log schemes:

tLogDecomp (2.2.5)

$$\begin{tikzcd} \mathrm{Spec}(P \to \mathbb{Z}[P]) \arrow[r] \arrow[d] & \mathrm{Spec}(M_0 \to \mathbb{Z}[M_0]) \arrow[d] \\ \mathrm{Spec}(M_0 \to \mathbb{Z}[M_0]) \arrow[r] & \mathrm{Spec}(M_1 \to \mathbb{Z}[M_1]) \end{tikzcd}.$$

Thus, the cartesian diagram in the statement of the lemma is obtained by pulling back the log structures of (2.2.5) via the diagram (2.2.4). ☐

s:LogStack

2.3. **Olsson's Log Stacks.** We follow [Ols03a] to introduce the algebraic stack parametrizing log schemes. Let us fix a base scheme S, and consider an algebraic stack $\mathcal{X}$ in the sense of [Art74], which means that

(1) the diagonal $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable and of finite type;
(2) there exists a surjective smooth morphism $X \to \mathcal{X}$ from a scheme.

Now we can define a fine log structure $\mathcal{M}_{\mathcal{X}}$ on $\mathcal{X}$ by repeating the definitions in 2.1.3 and 2.1.4 but using lisse-étale site instead of the étale site. See [Ols03a, Section 5] for details.

For any S-scheme T, and an arrow $g : T \to \mathcal{X}$, we obtain a fine log structure $g^*(\mathcal{M}_{\mathcal{X}})$ on the lisse-étale site $T_{lis\text{-}et}$ of T. It is shown in [Ols03a, 5.3] that such $g^*(\mathcal{M}_{\mathcal{X}})$ is isomorphic to a unique fine log structure on the étale site T_{et} of T. By abusing of notations, we still use $g^*(\mathcal{M}_{\mathcal{X}})$ denote this new log structure on T. By pulling back the log structure $\mathcal{M}_{\mathcal{X}}$, we define a functor from $\mathcal{X}$ to the category of fine log schemes over S. The stack $\mathcal{X}$ associated with this functor is called a log stacks in [Kat00]. A fine log scheme $(X, \mathcal{M}_X)$ can be naturally viewed as a log algebraic stack.

Consider the fibered category $\mathcal{L}og_{(\mathcal{X},\mathcal{M}_{\mathcal{X}})}$ over $\mathcal{X}$. Its objects are pairs $(g : X \to \mathcal{X}, g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X)$, where g is a map from scheme X to $\mathcal{X}$, and $g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X$ is a morphism of fine log structures on X. An arrow

$$\big(g : X \to \mathcal{X}, g^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_X\big) \longrightarrow \big(h : Y \to \mathcal{X}, h^*(\mathcal{M}_{\mathcal{X}}) \to \mathcal{M}_Y\big)$$

is a strict morphism of log schemes $(X, \mathcal{M}_X) \to (Y, \mathcal{M}_Y)$, such that the underlying map $X \to Y$ is a morphism over $\mathcal{X}$, and we have the following commutative diagram of log schemes:

$$\begin{tikzcd} (X, \mathcal{M}_X) \arrow[r] \arrow[d] & (Y, \mathcal{M}_Y) \arrow[d] \\ \big(X, g^*(\mathcal{M}_\mathcal{X})\big) \arrow[r] & \big(Y, h^*(\mathcal{M}_\mathcal{X})\big). \end{tikzcd}$$

Remark 2.17. In fact, an object $\big(g : X \to \mathcal{X}, g^*(\mathcal{M}_\mathcal{X}) \to \mathcal{M}_X\big)$ can be viewed as a morphism of log stacks $(X, \mathcal{M}_X) \to (\mathcal{X}, \mathcal{M}_\mathcal{X})$. Roughly speaking, the stack $\mathcal{L}og_{(\mathcal{X},\mathcal{M}_\mathcal{X})}$ parametrizes log schemes over $(\mathcal{X}, \mathcal{M}_\mathcal{X})$. For the definition of morphisms of log stacks, we refer to [Ols03a], and this one is compatible with the definition of morphisms between log schemes.

m:LogStack **Theorem 2.18.** [Ols03a, 5.9] *The fibered category $\mathcal{L}og_{(\mathcal{X},\mathcal{M}_\mathcal{X})}$ is an algebraic stack locally of finite presentation over $\mathcal{X}$.*

3. Logarithmic curves and their stacks

In this section, we define log pre-stable curves in our sense, and show that the stack $\mathfrak{M}_{g,n}^{pre}$ parametrizing log pre-stable curves of genus g and n marked points in our sense is an open substack of some Olsson's log stack as above, hence is algebraic in the sense of [Art74, 5.1].

3.1. **The canonical log structure on pre-stable curves.** We first introduce the canonical log structure on pre-stable curves. For details, we refer the reader to [Kat00], [S.M95], and [Ols07].

Let $\mathfrak{M}_{g,n}$ be the stack parametrizing genus g pre-stable curves with n marked points, and let $\mathfrak{C}_{g,n}$ be the universal family over $\mathfrak{M}_{g,n}$. Denote by $\{\Sigma_i : \mathfrak{M}_{g,n} \to \mathfrak{C}_{g,n}\}_{i=1}^n)$ the n sections. The boundary $\mathfrak{M}_{g,n}^{sing} \subset \mathfrak{M}_{g,n}$ which parametrizes singular curves is a divisor with normal crossings on $\mathfrak{M}_{g,n}$. Hence the boundary divisor induces a canonical log structure $\mathcal{M}_{\mathfrak{M}_{g,n}}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}$ on $\mathfrak{M}_{g,n}$, which is defined on the smooth topology in the sense of [Ols03a]. Note that the n sections $\{\Sigma_i\}$ and the pre-image of $\mathfrak{M}_{g,n}^{sing}$ in $\mathfrak{C}$ also give divisors with normal crossings on $\mathfrak{C}_{g,n}$, which induces another log structure $\mathcal{M}_{\mathfrak{C}}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}$ on $\mathfrak{C}_{g,n}$. There is a natural log smooth map $(\mathfrak{C}_{g,n}, \mathcal{M}_{\mathfrak{C}_{g,n}}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}) \to (\mathfrak{M}_{g,n}, \mathcal{M}_{\mathfrak{M}_{g,n}}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}})$ whose underlying map is given by the family $\mathfrak{C}_{g,n} \to \mathfrak{M}_{g,n}$.

Given any family $C \to S$ of usual pre-stable curves of genus g, with n marked points, we have the following cartesian diagram:

$$\begin{tikzcd} C \arrow[r] \arrow[d, "\pi"'] & \mathfrak{C}_{g,n} \arrow[d] \\ S \arrow[r] & \mathfrak{M}_{g,n}. \end{tikzcd}$$

Pulling back the canonical log structures on $\mathfrak{C}_{g,n}$ and $\mathfrak{M}_{g,n}$, we obtain canonical log structures $\mathcal{M}_C^{C/S}$ and $\mathcal{M}_S^{C/S}$ on C and S respectively, and a natural log smooth map $\pi : (C, \mathcal{M}_C^{C/S}) \to (S, \mathcal{M}_S^{C/S})$.

Using the notation as above, the log structure $\mathcal{M}_{\mathfrak{M}_{g,n}}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}$ is locally free, hence the canonical log structure $\mathcal{M}_S^{C/S}$ is also locally free. Then for any closed point $s \in S$, we have $\overline{\mathcal{M}}_{S,\bar{s}}^{C/S} \cong \mathbb{N}^m$, and this m equal to the number of the nodes in the fiber $C_{\bar{s}}$. In fact we have a one-to-one correspondence between the m factors of the monoid $\mathbb{N}^m$ and the nodes on the fiber.

s:LocalCan

3.2. **Local description of the canonical log structure on pre-stable curves.** By [Ols03a, 2.1], we can shrink S if necessary, and assume that we have a global chart $\mathbb{N}^m \to \mathcal{M}_S^{C/S}$ given by $\overline{\mathcal{M}}_{S,\bar{s}}^{C/S}$. We denote $\{e_i\}_{i=1}^m$ be the standard generators of $\mathbb{N}^m$.

Consider a closed point $p \in C_{\bar{s}}$ in the fiber. If p is a smooth non-marked point, then we have an étale neighborhood $\bar{p} \in U \subset C$, such that $\mathcal{M}_C^{C/S}|_U = \pi^*(\mathcal{M}_S^{C/S})|_U$.

When p is a marked point given by the section Σ_i, then consider an étale neighborhood $p \in U$ which contains only smooth points of C over S, and no other markings. We have the log structure

$$\mathcal{M}_C^{C/S}|_U = \pi^*(\mathcal{M}_S^{C/S})|_U \oplus_{\mathcal{O}_U^*} \mathcal{M}^{\Sigma_i}|_U,$$

where the log structure $\mathcal{M}^{\Sigma_i}$ is given by the section Σ_i, which locally has a chart $\mathbb{N} \to \mathcal{M}^{\Sigma_i}$. Hence we have a chart $\mathbb{N}^m \oplus \mathbb{N} \to \mathcal{M}_C^{C/S}|_U$.

Finally, let us assume p is a node. Then there is an étale neighborhood U of $\bar{p}$, which contains no other nodes and marked points. We have a special element $e_j \in \{e_i\}_{i=1}^m$, with the following chart:

$$\begin{tikzcd}
\mathbb{N}^{m-1} \oplus \mathbb{N}^2 \arrow[r] & \mathcal{M}_C^{C/S}|_U \\
\mathbb{N}^{m-1} \oplus \mathbb{N} \arrow[r] \arrow[u, "(id,\Delta)"] & \pi^*(\mathcal{M}_S^{C/S})|_U. \arrow[u, "\pi^\flat"']
\end{tikzcd}$$

Here on the bottom, the monoids $\mathbb{N}^{m-1}$ and $\mathbb{N}$ are generated by $\{e_i\}_{i \neq j}$ and e_j respectively, and on the top we assume that a and b are the standard generators of the monoid $\mathbb{N}^2$. The map (id, Δ) is given by the identity on $\mathbb{N}^{n-1}$ and the diagnonal map $\Delta : e_j \mapsto a + b$.

Definition 3.1. We identify e_j with its image in the log structre, and call it an element in $\mathcal{M}_S^{C/S}$ smoothing the node p, or simply an element smoothing p.

Note that two elements smoothing a same node are differ by an invertible function near the node, therefore they induce the same element in the characteristic monoid $\overline{\mathcal{M}}_S^{C/S}$.

For each node p_i over s, we fix an element e_i smoothing it. Denote by $\bar{e}_i$ the image of e_i in $\overline{\mathcal{M}}_S^{C/S}$. Let $Irr(\overline{\mathcal{M}}_{S,\bar{s}})$ be the set of irreducible elements in the monoid $\overline{\mathcal{M}}_{S,\bar{s}}$. In fact we have $\{\bar{e}_i\}_{i=1}^m = Irr(\overline{\mathcal{M}}_{S,\bar{s}})$, and a natural map:

$$s_{C_{\bar{s}}} : \{\text{nodes in } C_{\bar{s}}\} \to Irr(\overline{\mathcal{M}}_{S,\bar{s}})$$

given by $p_i \mapsto$ (the element e_i smoothes p_i). It was shown in [Kat00] that this map is a one-to-one correspondance. This means that all nodes in the fiber are smoothed independently.

em:Special

Remark 3.2. The bijection $s_{C_{\bar{s}}}$ implies that the canonical log structures $(\mathcal{M}_S^{C/S}, \mathcal{M}_C^{C/S})$ is special in the sense of [Ols03b, 2.6].

ode-To-Log

Remark 3.3. The one to one correspondance $s_{C_{\bar{s}}}$ associates to each node p_i a unique sub-log structure $\mathcal{N}_i \subset \mathcal{M}_S^{C/S}$ generated by e_i. In an étale neighborhood of $\bar{s}$, it was shown in [Kat00] that

$$\mathcal{M}_S^{C/S} \cong \mathcal{N}_1 \oplus_{\mathcal{O}_S^*} \cdots \oplus_{\mathcal{O}_S^*} \mathcal{N}_m.$$

CanLog

3.3. **The canonical log structure at node.** We give a local description of the relation between canonical log structure and the underlying structure at the nodes as in [Kat00, Section 3]. Let A be a local neotherian henselian ring, and s an element in the maximal ideal m_A of A. Let R be the henselization of $A[x,y]/(xy-s)$ at the ideal generated by x,y and m_A. We still use x,y to denote the corresponding elements in R.

erDesCurve

Lemma 3.4. *With the notation as above, we have the following:*

(1) [Kat00, 2.1] *Given $x',y' \in R$ such that $x'y' \in A$ and $(x',y',m_A) = (x,y,m_A)$ (equality of ideals in R). Then there exist units $u_x, u_y \in R^*$ with $u_x u_u \in A$ such that $x' = u_x x$ and $y' = u_y y$ (or $y' = u_x x$ and $x' = u_y y$).*

(2) [Kim, 3.6.1(2)] *Suppose that $x^c = u_x x^c$ and $y^c = u_y y^c$, where $c \in \mathbb{N}_{\geq 1}$ and $u_x, u_y \in R^*$. If $u_x u_y \in A^*$, then $u_x = u_y = 1$.*

Consider the local family $\mathrm{Spec}R \to \mathrm{Spec}A$, the canonical log structure $(\mathcal{M}_R, \mathcal{M}_A)$ is given by the following commutative diagram of prelog structures.

$$\begin{tikzcd}
\mathbb{N}^2 \arrow[r, "{(e_1,e_2)\mapsto(x,y)}"] & R \\
\mathbb{N} \arrow[u, "\Delta"] \arrow[r, "e\mapsto s"] & A \arrow[u]
\end{tikzcd}$$

where e_1, e_2 (resp. e) are the standard generators of $\mathbb{N}^2$ (resp. $\mathbb{N}$), and $\Delta : e \mapsto e_1 + e_2$ is the diagonal map. For convenience, we sometimes use $\log x, \log y$ and $\log s$ denote the image of e_1, e_2 and e in the corresponding log structures.

CanCurvGen

Corollary 3.5. [Kim, 3.6.2] *We use the notations as above, and let c be a positive integer. Then there is a unique pair γ_x, γ_y in $\mathcal{M}_R$, which will be denoted by $l\log x, l\log y$ respectively, such that $\gamma_x + \gamma_y \in \mathcal{M}_A$ and $\exp(\gamma_x) = x^l, \exp(\gamma_y) = y^l$*

ss:UnivCan

3.4. **Universal property of canonical log structure.** Next we introduce another description of the canonical log structre. In fact, this is the description given in [Kat00] and [Ols07, 3.9,3.10], except that in our case, we do not introduce orbifold structure.

Now we consider a new log structure on the fiber $\mathcal{M}_C^{\sharp C/S}$ which is obtained by removing the log structure corresponding to the markings. This is equivalent to require that the log structure near the marked points is pull back of the log structures from the base. By our description of canonical log structures, we have the relation

$$\mathcal{M}_C^{C/S} = \mathcal{M}_C^{\sharp C/S} \oplus_{\mathcal{O}_C^*} (\sum_j \mathcal{M}^{\Sigma_j}).$$

And we still have a log map $\pi^\sharp : \mathcal{M}_C^{\sharp C/S} \to \mathcal{M}_S^{C/S}$. This map is log smooth, proper, integral, vertical, and special (see remark 3.2). In fact, we have the following universal property.

UnivCanLog

Lemma 3.6. *For any pair of fine log structures $(\mathcal{M}'_C, \mathcal{M}_S)$ over the family of prestable curves $C \to S$, such that the log map $(C, \mathcal{M}'_C) \to (S, \mathcal{M}_S)$ is log smooth, proper, integral and vertical, we have a unique pair of maps $\mathcal{M}_C^{\sharp C/S} \to \mathcal{M}'_C$ and $\mathcal{M}_S^{C/S} \to \mathcal{M}_S$ fitting in the* *following cartesian diagram of fine log schemes:*

$$\begin{tikzcd}
(C, \mathcal{M}'_C) \arrow[r] \arrow[d] & (C, \mathcal{M}_C^{\sharp C/S}) \arrow[d] \\
(S, \mathcal{M}_S) \arrow[r] & (S, \mathcal{M}_S^{C/S}),
\end{tikzcd}$$

Proof. See [Ols07], and [Ols03b, 2.7] for a proof.

Remark 3.7. We remark that the canonical log structure $\mathcal{M}_S^{\sharp C/S}$ does not depend on the markings.

3.5. **Log curves.** With the description above, we are able to introduce the log structure on curves that we are interested in.

DefLogC1 **Definition 3.8.** A map of fine log schemes $(C, \mathcal{M}_C) \to (S, \mathcal{M}_S)$ with sections $\{\Sigma_i\}_{i=1}^n$ is called a genus g log curve with n-markings if

(1) the family $C \to S$ with $\{\Sigma_i\}$ is the usual prestable curve of genus g and n-markings;
(2) the log structure $\mathcal{M}_C$ is of the form $\mathcal{M}_C = \mathcal{M}'_C \oplus_{\mathcal{O}_C^*} (\sum_j \mathcal{M}^{\Sigma_j})$;
(3) the log map $(C, \mathcal{M}_C) \to (S, \mathcal{M}_S)$ comes from a log smooth, integral vertical map $(C, \mathcal{M}'_C) \to (S, \mathcal{M}_S)$ plus the log structure $\mathcal{M}^{\Sigma_i}$ given by the markings.

By lemma 3.6, we have an equivalent definition of log curves using the canonical log structure.

DefLogC2 **Definition 3.9.** A genus g, log curve with n-marked points over a scheme S is given by the following data $(C \to S, \{\Sigma\}_{i=1}^n, \mathcal{M}_S^{C/S} \to \mathcal{M}_S)$, where

(1) $(C \to S, \{\Sigma\}_{i=1}^n)$ is a usual family of pre-stable curves of genus g, n-markings;
(2) $\mathcal{M}_S^{C/S} \to \mathcal{M}_S$ is a morphism of fine log structures.

When no confusion would arise, we denote $(C \to S, \mathcal{M}_S)$ to be the log curves in the definition for short. We use $\mathcal{M}_C$ for the log structure on the curves in the above definition 3.8.

FrLogCurve

3.6. **Log pre-stable curves.**

ogPreCurve **Definition 3.10.** A log curve $(C \to S, \mathcal{M}_S)$ is called log pre-stable if the log structure $\mathcal{M}_S$ is fine and saturated.

For simplicity, we consider the case where S is a geometric point. Note that we have a map on the level of characteristic $\overline{\mathcal{M}}_S^{C/S} \to \overline{\mathcal{M}}_S$. Since the log structure $\overline{\mathcal{M}}_S^{C/S}$ is locally free, we fix $\overline{\mathcal{M}}_S^{C/S} \cong \mathbb{N}^m$, and denote by $\{e_i\}_{i=1}^m$ the set of all irreducible elements in $\overline{\mathcal{M}}_S^{C/S}$.Consider the map on the level of characteristic $\bar{\psi} : \overline{\mathcal{M}}_S^{C/S} \to \overline{\mathcal{M}}_S$. By remark 3.3, let p be the node corresponds to e_i. We call $\bar{\psi}(e_i)$ *the element smoothes p in* $\overline{\mathcal{M}}_S$. Later for convenience, we will identify e_i with its image $\bar{\psi}(e_i)$ in $\overline{\mathcal{M}}_S$.

CurveOpen **Remark 3.11.** By [Ols03a, 5.26], the condition that the base log structure $\mathcal{M}_S$ is fine and saturated is an open condition on S.

3.7. **The stack of log curves.**

soLogCurve **Definition 3.12.** Given two log curves $(C \to S, \mathcal{M}_S)$ and $(C' \to S, \mathcal{M}'_S)$ over S. Denote by $\mathcal{M}_C$ and $\mathcal{M}_{C'}$ the log structure on C and C' associated to the two log curves respectively. An isomorphism between the above two log curves is a pair (ρ, θ) such that

(1) $\theta : (S, \mathcal{M}_S) \to (S, \mathcal{M}'_S)$ and $\rho : (C, \mathcal{M}_C) \to (C', \mathcal{M}_{C'})$ are isomorphisms of log schemes;
(2) the underlying map $\underline{\theta} : S \to S$ is the identity, and $\underline{\rho} : C \to C'$ is an isomorphism of usual prestable curves over S;
(3) the pair (ρ, θ) fit in the following commutative diagram:

$$\begin{tikzcd}
(C, \mathcal{M}_C) \arrow[r, "\rho"] \arrow[d] & (C', \mathcal{M}_{C'}) \arrow[d] \\
(S, \mathcal{M}_S) \arrow[r, "\theta"] & (S, \mathcal{M}'_S).
\end{tikzcd}$$

Denote by $\mathfrak{M}^{log}_{g,n}$ the fibered category over $\mathbb{C}$ parametrizing log curves with the arrow defined above. In fact, we have

$$\mathfrak{M}^{log}_{g,n} \cong \mathcal{L}og_{(\mathfrak{M}_{g,n}, \mathcal{M}^{\mathfrak{C}_{g,n}/\mathfrak{M}_{g,n}}_{\mathfrak{M}_{g,n}})}.$$

Thus, the fibered category $\mathfrak{M}^{log}_{g,n}$ forms an algebraic stack in the sense of [Art74]. Denote by $\mathfrak{M}^{pre}_{g,n}$ the substack of $\mathfrak{M}^{log}_{g,n}$ parametrizing log prestable curves. Then by remark 3.11, we have the following:

CurveStack **Corollary 3.13.** *The fibered category $\mathfrak{M}^{pre}_{g,n}$ is an open substack in $\mathfrak{M}^{log}_{g,n}$, hence is algebraic.*

4. Algebricity of the stack of log maps

4.1. **Setup of notations.**

Conventions 4.1. In this section, we fix a projective, integral morphism of log schemes $\pi : X^{log} \to B^{log}$. Denote by B and X the underlying schemes of B^{log} and X^{log} respectively. Let $\mathcal{M}_B$ and $\mathcal{M}_X$ be the log structure on B^{log} and X^{log} respectively. Given any B-scheme S, Denote by $(X_S, \mathcal{M}^{X_S/S}_{X_S}) \to (S, \mathcal{M}^{X_S/S}_S)$ the pull-back of $X^{log} \to B^{log}$ over S.

efn:LogMap **Definition 4.2.** A log map over a B-scheme S is given by the datum

$$\xi = (C \to S, \pi_S : X_S \to S, \mathcal{M}^{X_S/S}_S \to \mathcal{M}_S, \mathcal{M}^{C/S}_S \to \mathcal{M}_S, f),$$

such that

(1) $(C \to S, \mathcal{M}_S)$ is a log curve;
(2) $\pi_S : X_S \to S$ fit in the following cartesian diagram of log schemes:

$$\begin{tikzcd}
(X_S, \mathcal{M}_X) \arrow[r] \arrow[d] & X^{log} \arrow[d, "\pi"] \\
(S, \mathcal{M}_S) \arrow[r] & B^{log}
\end{tikzcd}$$

(3) $f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$ is a log map over $(S, \mathcal{M}_S)$.

Given another B-scheme T, and a B-scheme morphism $g : T \to S$. The pull-back ξ_T of ξ via g is a log map over T, given by the following datum

$$(C_T \to T, X_T \to T, \mathcal{M}_T^{X_T/T} \to \mathcal{M}_T, \mathcal{M}_T^{C_T/T} \to \mathcal{M}_T, f_T)$$

where

(1) The underlying families $C_T \to T$ and $X_T \to T$ are the pull-back of the familis $C \to S$ and $X_S \to S$ via g respectively.
(2) The morphisms of log structures $\mathcal{M}_T^{X_T/T} \to \mathcal{M}_T$ and $\mathcal{M}_T^{C_T/T} \to \mathcal{M}_T$ are the pull-back of the morphisms $\mathcal{M}_S^{X_S/S} \to \mathcal{M}_S$ and $\mathcal{M}_S^{C/S} \to \mathcal{M}_S$ via g respectively.
(3) The log map f_T is the pull-back of f via the strict log map $(T, \mathcal{M}_T) \to (S, \mathcal{M}_S)$ induced by g.

In the following, if no confusion would arise, we will use $(C \to S, X_S \to S, \mathcal{M}_S, f)$ to denote the log map ξ over S.

:LogMapIso **Definition 4.3.** Consider two log maps $\xi_1 = (C_1 \to S, X_S \to S, \mathcal{M}_1, f_1)$ and $\xi_2 = (C_2 \to S, X_S \to S, \mathcal{M}_2, f_2)$ over S. An arrow $\xi_1 \to \xi_2$ over S is given by a triple (ρ, θ, γ) where

(1) The pair (ρ, θ) is an arrow of log curves $(C_1 \to S, \mathcal{M}_1) \to (C_2 \to S, \mathcal{M}_2)$ as in definition 3.12.
(2) The log map $\gamma : (X_S, \mathcal{M}_{X,1}) \to (X_S, \mathcal{M}_{X,2})$ is an isomorphism of log schemes fitting in the following commutative diagram:

:TargetIso (4.1.1)
$$\begin{tikzcd}
(X_S, \mathcal{M}_{X,1}) \arrow[rr, "\gamma"] \arrow[dr] \arrow[dd] & & (X_S, \mathcal{M}_{X,2}) \arrow[dl] \arrow[dd] \\
& X^{log} \arrow[dd] & \\
(S, \mathcal{M}_1) \arrow[rr, crossing over, "\theta"] \arrow[dr] & & (S, \mathcal{M}_2) \arrow[dl] \\
& B^{log} &
\end{tikzcd}$$

where the three squares are cartesian.
(3) The triple (ρ, θ, γ) fits in the following commutative diagram:

:LogMapIso (4.1.2)
$$\begin{tikzcd}
(C_1, \mathcal{M}_{C,1}) \arrow[rr, "f_1"] \arrow[dr] \arrow[dd, "\rho"] & & (X_S, \mathcal{M}_{X,1}) \arrow[dl] \arrow[dd, "\gamma"] \\
& (S, \mathcal{M}_1) \arrow[dd] & \\
(C, \mathcal{M}_{C,2}) \arrow[rr, crossing over, "f_2"] \arrow[dr] & & (X, \mathcal{M}_{X,2}) \arrow[dl] \\
& (S, \mathcal{M}_2) &
\end{tikzcd}$$

Note that under the above assumption, the underlying maps $\underline{\theta}$ and $\underline{\gamma}$ are identities. Denote by $\mathcal{I}som_S(\xi_1,\xi_2)$ the funtor over S, which for any S-scheme $T\to S$ associates the set of isomorphisms of $\xi_{T,1}$ and $\xi_{T,2}$ over T, where $\xi_{T,1}$ and $\xi_{T,2}$ are the pull-back of ξ_1 and ξ_2 via $T\to S$ respectively. Denote by $\mathcal{A}ut_S(\xi)$ the funtor of automorphisms of ξ over S.

Definition 4.4. Denote by $\mathcal{K}^{log}_{n,g}(X^{log}/B^{log})$ the fibered category over the category of B-schemes, such that for any $S\to B$, it associates the category of log maps over S, such that the underlying prestable curve is genus g, with n marked points. For simplicity, in this section we will use $\mathcal{K}^{log}$ to denote $\mathcal{K}^{log}_{n,g}(X^{log}/B^{log})$.

Denote by $\mathfrak{M}_{n,g}$ the algebraic stack of genus g, n-marked pre-stable curves with the canonical log structure. Consider the new algebraic stack

$$\mathfrak{B}=\mathcal{L}og_{\mathfrak{M}_{n,g}\times_{Log}B^{log}},$$

where the fibered product are in the log sense. Clearly $\mathfrak{B}$ is an algebraic stack over B.

ModuliBase **Remark 4.5.** We explain the moduli interpretation of $\mathfrak{B}$. For any B-scheme S, an object $\zeta_i\in\mathfrak{B}(S)$ is a diagram

iag:TarSou (4.1.3)
$$\begin{tikzcd}
(C_i,\mathcal{M}_{C_i}) \arrow[dr] & & (X_S,\mathcal{M}_{X_S,i}) \arrow[dl] \\
 & (S,\mathcal{M}_i) &
\end{tikzcd}$$

where the left arrow is a family of genus g, n-marked log curves given by the induced map $(S,\mathcal{M}_S)\to\mathfrak{M}_{n,g}$, and the right arrow is given by the induced map $(S,\mathcal{M}_S)\to B^{log}$. An arrow between two objects ζ_1 and ζ_2 is a triple (ρ,θ,γ) given by the following diagram

:IsoTarSou (4.1.4)
$$\begin{tikzcd}
(C_1,\mathcal{M}_{C,1}) \arrow[dr] \arrow[dd, "\rho"'] & & (X_S,\mathcal{M}_{X,1}) \arrow[dl] \arrow[dd, "\gamma"] \\
 & (S,\mathcal{M}_1) \arrow[dd, "\theta"'] & \\
(C,\mathcal{M}_{C,2}) \arrow[dr] & & (X,\mathcal{M}_{X,2}) \arrow[dl] \\
 & (S,\mathcal{M}_2) &
\end{tikzcd}$$

where the square on the left is an isomorphism of log curves, and the square on the right satisfies the condition in definition 4.3(2).

lateToBase **Remark 4.6.** Note that there is natural map $\mathcal{K}^{log}\to\mathfrak{B}$ by removing the log maps. It is not hard to see that this arrow is representable.

We denote by $\mathcal{K}_{n,g}(X/B)$ the stack of usual maps with the source genus g, n-marked pre-stable curves. This is an algebraic stack over B. For simplicity, we use $\mathcal{K}$ to denote this stack.

mpMapStack **Remark 4.7.** Note that we have a natural arrow $\mathcal{K}^{log}\to\mathcal{K}$ by removing all log structures. Given a log map ξ, denote by $\underline{\xi}$ the corresponding object in $\mathcal{K}$.

Our main result of this section is the following:

tackLogMap **Theorem 4.8.** *The fibered category $\mathcal{K}^{log}$ is an algebraic stack.*

Proof. The rest of this section is devote to the proof of this theorem. The representability of the diagonal $\mathcal{K}^{log} \to \mathcal{K}^{log} \times \mathcal{K}^{log}$ is proved in subsection 4.2. By remark 4.6, we have a natural representable map $\mathcal{K}^{log} \to \mathfrak{B}$ to the algebraic stack $\mathfrak{B}$. Thus, to produce a smooth cover for $\mathcal{K}^{log}$ is enough to check Artin's criteria [Art74, 5.1] relative to $\mathfrak{B}$. This will be done from subsection 4.3 to 4.7. □

ss:DiagRep

4.2. **Representability of the isomorphism functors of log maps.**

pIsoLogMap **Proposition 4.9.** *Consider two log maps ξ_1 and ξ_2 over a B-scheme S as in definition 4.3. The functor $\mathcal{I}som_S(\xi_1, \xi_2)$ is represented by an algebraic space locally of finite type over S.*

Proof. Using the notations as in definition 4.3, remark 4.5 and remark 4.7, we form the following commutative diagram:

IsoRelToBK (4.2.1)

$$\begin{tikzcd}
\mathcal{I}som_S(\xi_1,\xi_2) \arrow[dr, "\phi_3"] \arrow[drr, "\phi_2"] \arrow[ddr, "\phi_1"'] & & \\
& I \arrow[r] \arrow[d] & \mathcal{I}som_S(\underline{\xi_1},\underline{\xi_2}) \arrow[d, "\psi_2"'] \\
& \mathcal{I}som_S(\zeta_1,\zeta_2) \arrow[r, "\psi_1"] & \mathcal{I}som_S(\underline{\zeta_1},\underline{\zeta_2}),
\end{tikzcd}$$

where the square is cartesian, and ϕ_3 is given by the universal property of fiber product. Note that any isomorphism of ξ_1 and ξ_2 induces trivial isomorphism of the underlying strucutre of the target $X_S \to S$. Thus, the sheaf $\mathcal{I}som_S(\underline{\zeta_1}, \underline{\zeta_2})$ is the isomorphism of the underlying curves. Since $\mathcal{I}som_S(\underline{\xi_1}, \underline{\xi_2})$, $\mathcal{I}som_S(\underline{\zeta_1}, \underline{\zeta_2})$, and $\mathcal{I}som_S(\zeta_1, \zeta_2)$ are represented by algebraic spaces locally of finite type over S, the sheaf I is also representable and locally of finite type. Hence it is enough to show that ϕ_3 is representable and locally of finite type.

Consider an S-scheme U, and an arrow $U \to I$ given by a pair (τ, λ), where

$$\tau \in \mathcal{I}som_S(\zeta_1, \zeta_2)(U) \quad \text{and} \quad \lambda \in \mathcal{I}som_S(\underline{\xi_1}, \underline{\xi_2})(U),$$

such that their induced elements in $\mathcal{I}som_S(\underline{\zeta_1}, \underline{\zeta_2})(U)$ coincide. Now we have a cartesian diagram :

$$\begin{tikzcd}
I' \arrow[r] \arrow[d] & \mathcal{I}som_S(\xi_1,\xi_2) \arrow[d] \\
U \arrow[r, "(\tau,\lambda)"'] & I.
\end{tikzcd}$$

Here I' is the sheaf over U which for any $V \to U$ associated a unital set $\{*\}$ if $(\tau, \lambda)_V$ induces an isomorphism between $\xi_{1,V}$ and $\xi_{2,V}$, and the empty set otherwise. Next we will show that $I' \to U$ is a locally closed immersion of finite type.

For simplicity, we assume $U = S$, denote by $\tau = (\rho, \theta, \gamma)$ as in definition 4.3. We need to show that the commutativity of the following diagram of log schemes is represented by a locally closed immersion of finite type:

$$\begin{tikzcd}
(C_1, \mathcal{M}_{C_1}) \arrow[r, "f_1"] \arrow[d, "\rho"'] & (X, \mathcal{M}_{X,1}) \arrow[d, "\gamma"'] \\
(C_2, \mathcal{M}_{C_2}) \arrow[r, "f_2"] & (X, \mathcal{M}_{X,2}).
\end{tikzcd}$$

Since the map τ already gives an isomorphism of the underlying structure, we only need to consider the commutativity of

LogCommute

$$\begin{tikzcd}
\mathcal{M}_{C_1} & f_1^*\mathcal{M}_{X,1} \arrow[l, "f_1^\flat"'] \\
\rho^*\mathcal{M}_{C_2} \arrow[u, "\rho^\flat"] & \rho^* \circ f_2^*\mathcal{M}_{X,2}. \arrow[l, "\rho^*\circ f_2^\flat"] \arrow[u, "\gamma^\flat"']
\end{tikzcd} \tag{4.2.2}$$

And our statement follows from the following lemma. $\square$

m:IsoFinIm

Lemma 4.10. *Notations as in the above proposition, the condition that diagram (4.2.2) commutes is represented by a quasi-compact locally closed immersion $Z \to S$.*

Proof. The commutativity of diagram (4.2.2) is equivalent to the equality

LogCommute

$$\rho^\flat \circ (\rho^* \circ f_2^\flat) = f_1^\flat \circ \gamma^\flat. \tag{4.2.3}$$

It was shown in [Ols03a, 3.6] that on the level of characteristic, the condition that the above equality holds is an open condition on the fiber curves C_1. Since $C_1 \to S$ is flat and proper, by shrinking S, we can assume that the equality (4.2.3) on the level of characteristic holds.

Locally at a point $\bar{p} \in C_1$ over $\bar{s} \in S$, we choose a chart $P \to \rho^* \circ f_2^*\mathcal{M}_{X,2}$. We identify elements in P with their image in log structure. Denote by $\{\delta_i\}$ the set of generators on P. Consider an element δ_i, locally we have

$$f_1^\flat \circ \gamma^\flat(\delta_i) = e_1 + \log h_1,$$

and

$$\rho^\flat \circ (\rho^* \circ f_2^\flat)(\delta_i) = \rho^\flat(e_2) + \log h_2 = \theta^\flat e_2 + \log(\rho^* h_2),$$

where h_1 and h_2 are local regular functions near $\bar{p}$ and $\rho^{-1}(\bar{p})$ respectively, and e_1 and e_2 are sections from $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively. Since the equality (4.2.3) holds on the level of characteristic, we can assume that

:Represent

$$\theta^\flat(e_2) = e_1 + \log q_1 \quad \text{and} \quad \log(\rho^* h_2) = \log h_1 + \log q_2, \tag{4.2.4}$$

where q_1 in an invertible section at point $\bar{s}$, and q_2 is an invertible section at $\bar{p}$.

We first claim that the condition that q_2 is given by a pull-back of sections locally near $\bar{s}$ is represented by a locally closed immersion on the base S. We consider the situation when p is a node, other cases can be proved similarly. Locally near p, the structure sheaf is of the form $R = \mathcal{O}_{S,\bar{s}}[x,y]/(x\cdot y - u)$, where $u \in \mathcal{O}_{X,\bar{s}}$. Consider the completion $\hat{R} = \mathcal{O}_{S,\bar{s}}[[x,y]]/(x\cdot y - u)$. The image of q_2 in $\hat{R}$ is given by

owerSeries

$$q_2 = a_0 + \sum_{i>0} a_i x^i + \sum_{j>0} b_j x^j, \tag{4.2.5}$$

where $a_i, b_j \in \mathcal{O}_{S,\bar{s}}$. Denote by $I = (a_i, b_j)_{i,j\geq 1}$ the ideal in $\mathcal{O}_{S,\bar{s}}$. Note that the power series (4.2.5) is an element in the henselization of R with respect to the point p. Thus, it lifts to some open neighborhood of p. The ideal I also lift to a open neighborhood of $\bar{s}$. Further shrinking S, the closed scheme Z' given by I represents the condition that q_2 is a section on the base. This proves the claim.

Now we can cover C_1 by finitely many étale open covers $\{U_t\}$, and apply the above argument on each open set. Since the family $C_1 \to S$ is proper and flat, by shrinking and restricting S to the locally closed sub-scheme Z', we can assume that

(1) the projection $U_t \to S$ is surjective;
(2) for each U_t and generator δ_i, the corresponding section q_2 as in equation (4.2.4) is an invertible section on the base S.

To satisfy the equation (4.2.3), it is equivalent to have $q_1 \cdot q_2^{-1} = 1$ for all U_t and δ_i. This gives a closed immersion $Z \to S$. Note that the number of generators of P is finite. This proves the statement. $\square$

FinTypeIso

Remark 4.11. If the three functors $\mathcal{I}som_S(\underline{\xi_1}, \underline{\xi_2})$, $\mathcal{I}som_S(\underline{\zeta_1}, \underline{\zeta_2})$, and $\mathcal{I}som_S(\zeta_1, \zeta_2)$ in diagram (4.2.1) are all of finite type, then the proof of lemma 4.10 shows that the functor $\mathcal{I}som_S(\xi_1, \xi_2)$ is also of finite type. This is the case when later we discuss log stable maps.

Next, we check the Artin's criteria [Art74, 5.1].

:StackGlue

4.3. $\mathcal{K}^{log}$ **is a stack under étale topology.** By [Art74, 1.1], or [GLB00, Definition 3.1], we need to prove the following:

(1) the isomorphism functor is a sheaf under étale topology;
(2) any étale descent datum for objects of $\mathcal{K}^{log}$ is effective.

Since the isomorphism functor is shown to be representable, hence is a sheaf under étale topology. For the second condition, let $\{S_i \to S\}_i$ be an étale covering of S, and $\xi_i \in \mathcal{K}^{log}(S_i)$ for each i. Assume that we have isomorphism $\phi_{ij} : \xi_i|_{S_i \times_S S_j} \to \xi_j|_{S_i \times_S S_j}$ for each pair (i, j), which satisfy the cocycle condition.

For any i, let ζ_i be the corresponding log curve and target as in remark 4.5 for ξ_i. Since such ζ_i is parametrized by the algebraic stack $\mathfrak{B}$, we can glue them together to obtain ζ over S, whose restriction to each S_i is ζ_i. By our assumption, étale locally we have log map from ζ given by ξ_i. Since log map can be glued étale locally, we can glue them to obtain a log map ξ whose restriction to each S_i is ξ_i. Note that if each ξ_i is log stable, then ξ is log stable as well.

FiniteType

4.4. $\mathcal{K}^{log}$ **is limit preserving.** [1] Consider

$$R = \varinjlim R_i,$$

where R_i is a direct system of neotherian rings. Denote by $S = \mathrm{Spec} R$ and $S_i = \mathrm{Spec} R_i$. By [Art74, Section 1], we need to show that the following map of groupoids is an equivalence of categories:

$$\varprojlim \mathcal{K}^{log}(S_i) \to \mathcal{K}^{log}(S)$$

Given a log map $\xi = (C \to S, X_S \to S, \mathcal{M}_S, f)$ in $\mathcal{K}^{log}(S)$. Since the stack $\mathfrak{B}$ is locally of finite type, we have the family $\zeta = (C \to S, X_S \to S, \mathcal{M}_S)$ coming from $\zeta_i = (C_i \to S_i, X_{S_i} \to S_i, \mathcal{M}_{S_i})$ over S_i for some i. Also notice that we have an induced map $S \to \mathcal{K}$ given by the underlying map. Since $\mathcal{K}$ is locally of finite type, the underlying map $\underline{f}$ is coming from $\underline{f}_{i'}$ over some $S_{i'}$. We pick up i_0 such that $i_0 > i$ and $i_0 > i'$.

[1] Check the essential surjectivity again.

It remains to consider the map of log structures $f^\flat : f^*\mathcal{M}_X \to \mathcal{M}_C$. We first introduce two stacks $\mathcal{L}^\Delta$ and $\mathcal{L}^\Lambda$ as in [Ols05, section 2].

Remark 4.12. Consider a scheme U over $\mathbb{Z}$. Objects in $\mathcal{L}^\Delta(U)$ are commutative diagrams of log structures on U of the following form

eltaLogStr

$$\begin{tikzcd} & \mathcal{M}_1 \arrow[dl] \arrow[dr] & \\ \mathcal{M}_2 \arrow[rr] & & \mathcal{M}_3. \end{tikzcd} \tag{4.4.1}$$

Objects in $\mathcal{L}^\Lambda$ are diagrams of log structures on U of the following form

mbdaLogStr

$$\begin{tikzcd} & \mathcal{M}_1 \arrow[dl] \arrow[dr] & \\ \mathcal{M}_2 & & \mathcal{M}_3. \end{tikzcd} \tag{4.4.2}$$

It was shown in [Ols05, 2.4] that those two stacks $\mathcal{L}^\Delta$ and $\mathcal{L}^\Lambda$ are algebraic stacks locally of finite type. Note that there is a natural morphism $\mathcal{L}^\Delta \to \mathcal{L}^\Lambda$ by dropping the bottom arrow in diagram (4.4.1) to obtain (4.4.2).

:LogStrMap

Remark 4.13. Consider $\zeta = (\pi_C : C \to S, X_S \to S, \mathcal{M}_S)$ the family of log sources and targets constructed above. There is a natural diagram of log structures on C as follows

LogOnCurve

$$\begin{tikzcd} & \pi_C^*\mathcal{M}_S \arrow[dl] \arrow[dr] & \\ f^*\mathcal{M}_X & & \mathcal{M}_C. \end{tikzcd} \tag{4.4.3}$$

This induces a natural map $C \to \mathcal{L}^\Lambda$. Consider the fiber product $\mathcal{L}^\Delta \times_{\mathcal{L}^\Lambda} C$. This gives an algebraic stack parametrizing the bottom arrows $f^\flat$ that fits in the above commutative diagram.

The map $f^\flat$ is equivalent to a map $C \to \mathcal{L}^\Delta \times_{\mathcal{L}^\Lambda} C$. Note that the algebraic stack $\mathcal{L}^\Delta \times_{\mathcal{L}^\Lambda} C$ is locally of finite presentation. By [GLB00, Proposition 4.18(i)], we have the map $f^\flat$ coming from some $f^\flat_{i_1}$ over S_{i_1} for some $i_1 > i_0$. This map is compatible with all the log structures coming from base and target. Indeed, consider the composition

$$p_j : C_j \to \mathcal{L}^\Delta \times_{\mathcal{L}^\Lambda} C_j \to C_j.$$

Applying [GLB00, Proposition 4.18(i)] again, we see that the identity $p = id_C : C \to C$ is coming from p_j for some $i_2 > i_1$. Thus, the map f_{i_2} also compatible with the underlying map $\underline{f}$. This proves the essential surjectivity.

The full faithfulness follows from [GLB00, Proposition 4.15(i)] and the fact that the diagonal $\mathcal{K}^{log} \to \mathcal{K}^{log} \times \mathcal{K}^{log}$ is representable and locally of finite type.

DefObs

4.5. **Deformations and obstructions.** By [Art74, Definition 5.1], it remains to find a smooth cover of $\mathcal{K}^{log}$. As in remark 4.6, we have a representable map of stack $\mathcal{K}^{log} \to \mathfrak{B}$. Since $\mathfrak{B}$ is an algebraic stack, it would be enough to produce a smooth cover for $\mathcal{K}^{log}_U := \mathcal{K}^{log} \times_{\mathfrak{B}} U$, where $U \to \mathfrak{B}$ is an arbitrary smooth map. This can be done by checking Artin's criteria [Art74, 5.2] for $\mathcal{K}^{log}_U$ relative to U. First we consider the deformations and obstructions.

Let A_0 be a reduced neotherian ring over U, and $A' \to A \to A_0$ be an infinitesimal extension of A_0, where $A' \to A$ is surjective whose kernel I is a finite A_0-module, hence is a square-zero ideal. Denote by $S = \mathrm{Spec}A$ and $S' = \mathrm{Spec}A'$. Consider a log map $\xi_A = (C \to S, X_S \to S, \mathcal{M}_S, f) \in \mathcal{K}_U^{log}$. Let $\xi_0 = (C_0 \to S_0, X_{S_0} \to S_0, \mathcal{M}_{S_0}, f_0)$ be the restriction of ξ_A over A_0. Since we are over U, the log source and target $(C \to S, X_S \to S, \mathcal{M}_S)$ come from the structure morphism $S \to U$. Note that we have another family of log source and target $(C' \to S', X_{S'} \to S', \mathcal{M}_{S'})$, which are also from the structure map $S_1 \to U$. To obtain a deformation of ξ_A over S' is equivalent to produce a dotted arrow f' that fits in the following log commutative diagram:

:DeformMap (4.5.1)

$$\begin{tikzcd}
(C,\mathcal{M}_C) \arrow[rr, "k"] \arrow[dr, "f"] \arrow[ddr, bend right] & & (C',\mathcal{M}_{C'}) \arrow[dr, dotted, "f'"] \arrow[ddr, bend right] & \\
& (X_S,\mathcal{M}_{X_S}) \arrow[rr, "j"] \arrow[d] & & (X_{S'},\mathcal{M}_{X_{S'}}) \arrow[d] \\
& (S,\mathcal{M}_S) \arrow[rr, "i"] & & (S',\mathcal{M}_{S'})
\end{tikzcd}$$

Note that the front and back squares in diagram (4.5.1) are cartesian of log schemes. Let $\mathbf{L}^{log}_{X_S/S}$ be the logarithmic cotangent complex of the log map $(X_S, \mathcal{M}_{X_S}) \to (S, \mathcal{M}_S)$ as in [Ols05]. By [Ols05, 5.9], we have the following results:

(1) there is a canonical class $o \in Ext^1(f^*\mathbf{L}^{log}_{X_S/S}, I \otimes_{A_0} \mathcal{O}_{C_0})$, whose vanishing is necessary and sufficient for the existence of a morphism f' fit into the above diagram.
(2) if $o = 0$, then the set of such maps f' is a torsor under $Ext^0(f^*\mathbf{L}^{log}_{X_S/S}, I \otimes_{A_0} \mathcal{O}_{C_0})$.

Thus we define $\mathcal{D}_{\xi_A}(I) = Ext^0(f^*\mathbf{L}^{log}_{X_S/S}, I \otimes_{A_0} \mathcal{O}_{C_0})$ and $\mathcal{O}_{\xi_A}(I) = Ext^1(f^*\mathbf{L}^{log}_{X_S/S}, I \otimes_{A_0} \mathcal{O}_{C_0})$ to be the module of deformations and obstructions. Note that the log cotangent complex $\mathbf{L}^{log}_{X_S/S}$ is bounded above with coherent cohomologies. The conditions of deformation and obstruction modules in [Art74, 5.2(4)] follows from the standard property of cohomology, see for example [AV02, 5.3.4].

ss:SchCond

4.6. **Schlessinger's conditions.** By [Art74, 5.2(2)], we need to verify Schlessinger's conditions (S1) and (S2) as in [Art74, section 2]. The condition (S2) follows from the cohomological description of the module of deformation $\mathcal{D}$. Next we check the condition (S1$'$) [Art74, 2.3], which is a stronger version of (S1).

Indeed, consider an infinitesimal extension $A' \to A \to A_0$ as in subsection 4.5, and a U-algebra homomorphism $B \to A$ such that the composition $B \to A_0$ is surjective. Consider $\xi_A \in \mathcal{K}_U^{log}(A)$. For any surjection $R \to A$, denote by $\mathcal{K}^{log}_{\xi_A}(R)$ the category of log maps over $\mathrm{Spec}R$ whose restriction to $\mathrm{Spec}A$ is ξ_A. Then we need to show that

$$\mathcal{K}^{log}_{\xi_A}(A' \times_A B) \to \mathcal{K}^{log}_{\xi_A}(A') \times \mathcal{K}^{log}_{\xi_A}(B)$$

is an equivalence of categories.

First, consider the essential surjectivity. Given objects $\xi_{A'} \in \mathcal{K}^{log}_{\xi_A}(A')$ and $\xi_B \in \mathcal{K}^{log}_{\xi_A}(B)$. Denote by $\xi_{A'} = (\zeta_{A'}, f_{A'})$ and $\xi_B = (\zeta_B, f_B)$, where $\zeta_{A'}$ and ζ_B are the corresponding log sources and targets as in remark 4.5. Since the two families $\zeta_{A'}$ and ζ_B correspond to maps

Spec$A' \to U$ and Spec$B \to U$, which induce the same map Spec$A \to U$ by restricting to Spec$A$. Then we can glue them togather to obtain Spec$B \times_A A' \to U$, and hence obtain a family $\zeta_{B\times_A A'}$ over Spec$B \times_A A'$, whose restrictions to SpecA' and Spec$B$ are $\zeta_{A'}$ and ζ_B respectively. Since the stack $\mathcal{K}$ parametrizing the underlying maps is algebraic, the same argument as above produces a gluing $\underline{f}_{A'\times_A B}$ of $\underline{f}_{A'}$ and $\underline{f}_B$.

It remains to produce a compatible morphism of log structures $f^\flat_{A'\times_A B}$. Next we choose an affine open cover $V_{B\times_A A'} = \bigcup_i V_i$ of the log source curve in $\zeta_{B\times_A A'}$, its restrictions to A' and B give the affine open covers V_B and V_A for curves of $\zeta_{A'}$ and ζ_B respectively. Consider the stack

$$\mathcal{L}^\Delta \times_{\mathcal{L}^\wedge} C_{A'} \quad \text{and} \quad \mathcal{L}^\Delta \times_{\mathcal{L}^\wedge} C_B,$$

induced by the log family $\zeta_{A'}$ and ζ_B respectively as in remark 4.13. They can be glued to give $\mathcal{L}^\Delta \times_{\mathcal{L}^\wedge} C_{A'\times_A B}$ which corresponds to $\zeta_{A'\times_A B}$. Consider the maps $V_{A'} \to \mathcal{L}^\Delta \times_{\mathcal{L}^\wedge} C_{A'}$ and $V_B \to \mathcal{L}^\Delta \times_{\mathcal{L}^\wedge} C_B$ induced by $f_{A'}$ and f_B respectively. Note that these maps can be glued together and descent to a map

$$C_{A'\times_A B} \to \mathcal{L}^\Delta \times_{\mathcal{L}^\wedge} C_{A'\times_A B}.$$

This induce a map of log structures

$$f^\flat_{A'\times_A B} : \underline{f}^*_{A'\times_A B}\mathcal{M}_{X_{A'\times_A B}} \to \mathcal{M}_{C_{A'\times_A B}}.$$

We can check that $f^\flat_{A'\times_A B}$ compatible with $\zeta_{A'} \times_A B$ and the underlying map $\underline{f}_{A'\times_A B}$.

The full faithfulness follows from the representability of isomorphism functor of log maps.

Completion

4.7. **Compatibility with formal completion.** Let $\hat{A}$ be a complete local ring, and m be the maximal ideal of $\hat{A}$. Denote by $A_n = \hat{A}/m^n$, $S = $ Spec$\hat{A}$, and $S_n = $ SpecA_n. Given a family of log maps $\{\xi_n = (C_n \to S_n, X_{S_n} \to S_n, \mathcal{M}_S, f_n)\}_n$ such that $\xi_n \in \mathcal{K}^{log}_U(S_n)$, and $\xi_n|_{S_k} = \xi_k$ for any $n \geq k$. According to [Art74, 5.2(3)], we need to show that there exists an element $\xi \in \mathcal{K}^{log}_U(S)$, such that $\xi|_{S_n} = \xi_n$ for any n.

Denote by $\zeta_n = (C_n \to S_n, X_{S_n} \to S_n, \mathcal{M}_{S_n})$ the family of log sources and targets of ξ_n. For each n, there is a map $S_n \to U$ induced by ζ_n, such that they fit in the following commutative diagrams for any $k \leq n$:

$$\begin{tikzcd} S_n \arrow[r] \arrow[d] & U \\ S_k \arrow[ur] & \end{tikzcd}$$

Note that the above diagram induces a map $S \to U$, whose restriction to S_n is the map given by ζ_n as above. Hence, we obtain a family of log sources and targets $\zeta = (C \to S, X_S \to S, \mathcal{M}_S)$ by pull-back the family of log curves over U. Note that $\zeta|_{S_n} = \zeta_n$ for any n.

Denote by $\underline{\xi_n}$ the usual prestable map over S_n. Consider the family of compatible underlying maps $\{\underline{\xi_n}\}$. By [GD61, 5.4.1], there exists a unique (up to a unique isomorphism) $\underline{f} : C \to X_S$ such that $\underline{f}|_{S_n} = \underline{f_n}$.

Now to construct ξ, we need to construct a log map $f : (C, \mathcal{M}_C) \to (X_S, \mathcal{M}_{X_S})$, which is compatible with the underlying map $\underline{f}$ and f_n for all n. By definition of log maps, this is equivalent to construct a map of log structures $f^\flat : \underline{f}^*\mathcal{M}_{X_S} \to \mathcal{M}_C$, which is compatible with $f^\flat_n$. For simplicity, denote by $\mathcal{M} = \underline{f}^*\mathcal{M}_{X_S}$.

To construct $f^\flat$, note that we have a family of maps $\{(f_n^*\mathcal{M}_{X_{S_n}})^{gp} \to \mathcal{M}_{C_n}^{gp}\}$ induced by $f_n^\flat$. Since we have $\mathcal{M}^{gp}|_{S_n} = (f_n^*\mathcal{M}_X)^{gp}$ and $\mathcal{M}_C^{gp}|_{S_n} = \mathcal{M}_{C_n}^{gp}$, by taking limit of sheaves of abelian groups, we obtain a map $\mathcal{M}^{gp} \to \mathcal{M}_C^{gp}$. Since we are working with fine log structures, we have an injection of sheaves $\mathcal{M} \to \mathcal{M}^{gp}$, then we have an induced map $\tilde{f} : \mathcal{M} \to \mathcal{M}_C^{gp}$. We first show that $Im(\tilde{f}) \subset \mathcal{M}_C \subset \mathcal{M}_C^{gp}$.

Assume on the contrary that there exists an étale open set $V \subset C$ and a section $a \in \Gamma(\mathcal{M}, V)$ such that $b = \tilde{f}(a) \notin \mathcal{M}_C|V$. Denote by $\pi^{gp} : \mathcal{M}_C^{gp} \to \overline{\mathcal{M}}_C^{gp}$ the canonical projection. Then $\pi^{gp}(b) \notin \overline{\mathcal{M}}_C|_V$. The closed points of C and C_n forms the same underlying topological space, write $\hat{C}$. We can view $\overline{\mathcal{M}}_C^{gp}$ and $\overline{\mathcal{M}}_C$ to be sheaves of groups and monoids on $\hat{C}$ respectively. Then we have $\overline{\mathcal{M}}_C = \overline{\mathcal{M}}_{C_n}$ and $\overline{\mathcal{M}}_C^{gp} = \overline{\mathcal{M}}_{C_n}^{gp}$. This implies that $\pi^{gp}(b)|_{C_n} \notin \overline{\mathcal{M}}_{C_n}$. But by our construction, $\tilde{f}(a)|_{C_n} = f_n^\flat(a) \in \mathcal{M}_{C_n}$, which implies $\pi^{gp}(b)|_{C_n} \in \overline{\mathcal{M}}_{C_n}$. This is a contradiction! Thus we obtain a well-defined map of sheaves of monoid $f^\flat : \mathcal{M} \to \mathcal{M}_C$, which is compatible with $f_n^\flat$.

To show that $f^\flat$ is map of log structures, it remains to show that the following diagram is commutative:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "f^\flat"] \arrow[d, "\alpha_1"'] & \mathcal{M}_C \arrow[dl, "\alpha_2"] \\
\mathcal{O}_C, &
\end{tikzcd}$$

where α_1 and α_2 are the structure morphism of the corresponding log structures. To see this, consider any section $s \in \mathcal{M}$. Since $\alpha_1(s)|_{S_n} = \alpha_1 \circ f^\flat(s)|_{S_n}$ for any n, we have $\alpha_1(s) = \alpha_1 \circ f^\flat(s)$. This proves the commutativity.

Finally, we need to show that $f^\flat$ is compatible with the log structure on the base. This is equivalent to show the commutativity of the following diagram of log structures on C:

$$\begin{tikzcd}
& \mathcal{M}_S \arrow[dl] \arrow[dr] & \\
f^*\mathcal{M}_{X_S} \arrow[rr, "f^\flat"] & & \mathcal{M}_S.
\end{tikzcd}$$

This can be checked using the functoriality of projective limit of groups, and the following commutative diagram for each n:

$$\begin{tikzcd}
& \mathcal{M}_{S_n} \arrow[dl] \arrow[dr] & \\
f^*\mathcal{M}_{X_{S_n}} \arrow[rr, "f^\flat"] & & \mathcal{M}_{S_n}.
\end{tikzcd}$$

Now the pair $(\underline{f}, f^\flat)$ gives the log map $f : (C, \mathcal{M}_C) \to (X_S, \mathcal{M}_{X_S})$ over $(S, \mathcal{M}_S)$, as we needed.

This finishes the proof of theorem 4.8. □

5. Logarithmic maps to Deligne-Faltings log pairs

efn:Target **Definition 5.1.** We call the log scheme $X^{log} = (X, \mathcal{M}_X)$ a *Deligne-Faltings log pair* or simply *a log pair*, if

(1) X is a projective variety;
(2) $\mathcal{M}_X$ is a DF log structure on X as in definition 2.9.

Conventions 5.2. In this section, we fix a log pair $(X, \mathcal{M}_X)$ as our target of log maps, with a global presentation $P \to \overline{\mathcal{M}}_X$, where P is a toric monoid as in (2.1.4). Denote by $Irr(P) = \{\delta_i\}_{i=1}^k$ the set of irreducible elements in P, and $\{\gamma_j\}_{j=1}^r$ the set of relations between the irreducible elements as in (2.1.4).

Note that each δ_i induces a rank one locally free sub-log structure $\mathcal{N}_i \subset \mathcal{M}_X$. Denote by (L_i, s_i) the line bundle and the global section corresponds to $\mathcal{N}_i$. Let D_i be the vanishing locus of the dual section $s_i^\vee \in H^0(L_i^\vee)$. By a nice choice of the global presentation, we require that D_i is non-empty and connceted for any i. We emphasis that this requirement is important for putting the contact orders, which we will discuss later.

Note that at each geometric point $\bar{p} \in X$, we have a surjective map of monoid $P \to \overline{\mathcal{M}}_{X,\bar{p}}$. For convenience, we identify δ_i with its image in $\overline{\mathcal{M}}_{X,\bar{p}}$.

Remark 5.3. Since D_i is connected, the set $\{D_i\}_{i=1}^k$ does not depent on the choice of P.

dleExplain **Remark 5.4.** Note that if $s_i = 0$, then $D_i = X$. In this case, the pair (L_i, s_i) gives a generic part $\mathcal{N}_i$ as in 2.13. If s_i is not a zero section, then D_i is a divisor in X. Thus, we have $L_i = \mathcal{O}_X(-D_i)$, and the section $s_i : \mathcal{O}_X(-D_i) \hookrightarrow \mathcal{O}_X$ is the natural inclusion. The section δ_i locally corresponds to a section in $\mathcal{O}_X$, whose vanishing locus gives the divisor D_i.

LogMapToDF **Remark 5.5.** Note that in the above case the target of the log maps is over a point with trivial log structures. Thus, we can simplify the notations as follows. A log map over S is given by the triple $(C \to S, \mathcal{M}_S, f)$, where $(C \to S, \mathcal{M}_S)$ is a log curve, and $f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X)$ is a log map. This is compatible with definition 4.2.

LogMapChar

5.1. **Log morphism on the level of characteristic.** Consider a log map $\xi = (\pi : C \to S, \mathcal{M}_S, f)$ as in definition 4.2, where $S = \mathrm{Spec} k$ is a geometric point and $(C \to S, \mathcal{M}_S)$ is a log prestable curve. Consider a point $p \in C$, which sits in an irreducible component Z. Then on the level of characteristic, we have a map

:CharMapSm (5.1.1)
$$\bar{f}_p^\flat : f^*(\overline{\mathcal{M}}_X)_p \to \overline{\mathcal{M}}_{C,p}.$$

First consider the case p is a smooth non-marked point. By the description in definition 3.8, we have $\bar{f}^\flat(\delta_i) = e_i \in \overline{\mathcal{M}}_S$. We call it *the i-th degeneracy at p*. By proposition 2.4, the smooth non-marked points in Z will all have the same i-th degeneracy. Thus, we call the element e_i *the i-th degeneracy of Z*.

Note that if $p \notin D_i$ for some $p \in Z$, then the image $e_i = 0 \in \overline{\mathcal{M}}_S$. Note that in this case, the component Z does not map to the divisor D_i.

Definition 5.6. The k-tuple $(e_i)_{i=1}^k$ is called *the degeneracy of Z*, where e_i is the i-th degeneracy of Z. Denote by $I_Z = \{\ i \mid e_i \neq 0\}$.

SmRelation **Remark 5.7.** Since (5.1.1) is a map of monoid, then the elements $\{e_i\}_{i=1}^k$ also satisfies the set of relations $\{\gamma_j\}_{j=1}^r$ by replacing δ_i with e_i. Consider the sub-monoid

$$M := \big\langle e_1, \cdots, e_k \mid \gamma_j, \text{for } j = 1, \cdots, r \big\rangle \subset \overline{\mathcal{M}}_{C,p}.$$

Since the map $P \to \overline{\mathcal{M}}_X$ locally liftes to a chart, it is not hard to check that the monoid M does not depend on the choice of global presentation $P \to \overline{\mathcal{M}}_X$.

Next, we consider the case where p is a marked point. Since locally at p, we have $\mathcal{M}_C \cong \pi^*\mathcal{M}_S \oplus_{\mathcal{O}^*_C} \mathcal{N}$, where $\mathcal{N}$ the the canonical log structure associated to the marked point p. Then on the level of characteristic, we have

u:MarkedPt
$$\bar{f}^\flat(\delta_i) = e_i + c_{i,p} \cdot \sigma_p, \tag{5.1.2}$$

where $e_i \in \overline{\mathcal{M}}_S$, and σ_p is the generator of $\overline{\mathcal{N}}$, and $c_{i,p}$ is a positive integer.

ngencyCond **Remark 5.8.** The same reason as in remark 5.7 shows that the set of elements $\{c_{i,p} \cdot \sigma_p\}$ satisfies the set of relations $\{\gamma_j\}_{j=1}^r$ by replacing δ_i with $c_{i,p} \cdot \sigma_p$.

egMarkedPt **Remark 5.9.** When we generalize the equation (5.1.2) to the nearby smooth points, the element σ_p will become ivertible in the structure sheaf. Thus, the element e_i is the i-th degeneracy of the component Z.

Definition 5.10. We call $c_{i,p}$ *the i-th contact order of f at* p, and the sum $c_p = \sum_{i=1}^k c_{i,p}$ the contact order at p.

ontactOpen **Lemma 5.11.** *Consider a log map $\xi = (C' \to S', \mathcal{M}_{S'}, g)$ over S'. There is an open subset in S', such that the fiber has the assigned contact order along a fixed marking.*

Proof. Since the contact order is a condition on the level of characteristic, by [Ols03a, 3.5], this is an open condition on the fiber. Since the source curve is flat and proper over the base, the assigned contact order is also an open condition on the base. □

Remark 5.12. When D_i is a divisor, the i-th contact order can be viewed as the tangency multiplicity with the divisor D_i.

Finally, let us consider the case where p is a node joining Z with another irreducible component Z'. Let e be the element in $\overline{\mathcal{M}}_S$ smoothing the node p, and $\log x_p, \log y_p$ are the elements in $\overline{\mathcal{M}}_C$ correspond to the local coordinates of the two components Z and Z' respectively as in subsection 3.3. Then we have the equation in $\overline{\mathcal{M}}_C$:

:NodeCurve
$$e = \log x_p + \log y_p. \tag{5.1.3}$$

Thus, without loss of generality we can assume that

equ:Node
$$\bar{f}^\flat(\delta_i) = e_i + c_{i,p} \cdot \log x_p, \tag{5.1.4}$$

where $c_{i,p}$ is a positive integer.

n:DistNode **Definition 5.13.** The integer $c_{i,p}$ is called *the i-th contact order of f at the node p.* If $c_{i,p} \neq 0$, then p is called an *i-distinguished node.*

DegNodeLow **Remark 5.14.** The same argument as in remark 5.9 shows that e_i is the degeneracy of Z. We call that Z is the i-lower component of p, and Z' is the i-upper component of p. This gives an ordering on the set of irreducible components.

ontactOpen **Lemma 5.15.** *Consider a log map $\xi = (C' \to S', \mathcal{M}_{S'}, g)$, and a connected singularity $p \subset C'$. There is an open subset in S', such that over each fiber we have that either the node p is smoothed out, or its i-th contact order remains the same.*

Proof. The proof is identical to the one for lemma 5.11.

em:DegJump **Lemma 5.16.** *Using notations as above, the i-th degeneracy of Z' is $e_i + c_{i,p} \cdot e$.*

Proof. Note that when generalize the equation 5.1.4 to a smooth point in Z', the section y becomes invertible. Then the statement follows from [Ols03a, 3.5(1)]. □

ss:AdGraph

5.2. **Admissible graph.**

eightGraph **Definition 5.17.** A *k-weighted graph* G is a connected graph with the following data:

(1) a set of vertices $V(G)$, such that for each $v \in V(G)$ we associate a k-tuple $(e_{v,i})_{i=1}^{k}$ called the weights of v, where $e_{v,i}$ is either 0 or a variable;
(2) A set of edges $E(G)$, such that for each $l \in E(G)$ we associate a k-tuple of non-negative integers $(c_{l,i})_{i=1}^{k}$ called the contact orders of l, and a variable e_l called the weight of l.

These data satisfies the only condition that if the edge l is a loop, then $c_{l,i} = 0$ for all i.

If the contact orders of an edge l are all zero, then l is called the non-distinguished edge. Two vertices is called adjacent if they are connected by an edge. Denote by $\underline{G}$ the underlying graph of G, obtained by removing all weights and contact orders.

rientation **Definition 5.18.** Consider a k-weighted graph G as in the above definition. An *k-orientation* on G is a set of k (possibly different) orientations on the underlying graph $\underline{G}$. Consider any two vertices v_1 and v_2 joined by an edge $l \in E(G)$. We write $v_1 \leq_i v_2$ if the edge l is from v_1 to v_2 under the i-th orientation for $i = 1, 2, \cdots, k$. Then these data should compatible with the weights on G as follows:

(1) If $c_{l,i} \neq 0$, then we have either $v_1 \leq_i v_2$ or $v_2 \leq_i v_1$;
(2) If $c_{l,i} = 0$, then we have both $v_1 \leq_i v_2$ and $v_2 \leq_i v_1$;
(3) If for a vertex v we have $e_{v,i} = 0$, then for any other adjacent vertex v' of v we have $v \leq_i v'$.

Consider an edge $l \in E(G)$, and its two end vertices v_1 and v_2. If $v_1 \leq_i v_2$, we call v_1 the i-initial vertex of l, and v_2 the i-end vertex of l. An i-path is a non-repeated squence of edges $(l_1, l_2, \cdots, l_m)$ such that the i-end vertex of l_j is the i-initial vertex of l_{j+1}. Such i-path is called an i-loop if the i-initial vertex of l_1 is the i-end vertex of l_m. A vertex $v \in V(G)$ is called i-minimal (respectively i-maximal) if it is not the i-end (respectively i-initial) vertex of any edge. Thus, any vertex v with the zero i-th weight is i-minimal.

Consider a k-weighted oriented graph G as in the above definition. For each edge $l \in E(G)$ and its i-initial vertex v_1 and i-end vertex v_2, we can associate an equation

u:GraphEqu
$$h_{l,i}: \quad e_{v_2,i} = e_{v_1,i} + c_{l,i} \cdot e_l \tag{5.2.1}$$

For each vertex v, we put a set of relations $\{\gamma_j\}_{j=1}^{r}$ as in (2.1.4), by replacing δ_i with $e_{v,i}$. Consider the monoid

arseMonoid
$$\tilde{M}(G) = \Big\langle e_{v,i}, e_l \,|\, \text{for all } i \in \{1, \cdots, k\}, \text{ and } l \in E(G), \text{ with all the relations } h_{l,i} \text{ and } \gamma_j \Big\rangle. \tag{5.2.2}$$

Denote by $M(G)$ the saturation of $\tilde{M}(G)$.

fn:AdGraph **Definition 5.19.** We call $M(G)$ the P-associated monoid of the k-weighted oriented graph G.

s:DegGraph **Construction 5.20.** For each k-weighted oriented graph G, we would like to focus on the non-zero weights. We associated a new graph G^{deg} as follows:

(1) Identify all vertices in G with only zero weights.
(2) Identify all vertices and their weights that are connected by a path formed by non-distinguished edges.

(3) Contract all non-distinguished edges.

By (2) above, we can define the weights of vertices in G^{deg} given by the weights from the corresponding vertices in G. The weights and contact orders of edges in G^{deg} can be obtained from the corresponding edges in G, since we only contract non-distinguished edges. Note the G^{deg} is a k-weighted graph.

Definition 5.21. The graph G^{deg} is called the *contracted graph of G*.

tractGraph **Proposition 5.22.** *Using the notations as above, we have*

(1) *The orientation $\{\leq_i\}$ in G induces an orientation in G^{deg}. Thus G^{deg} is a k-weighted oriented graph.*
(2) *We have a canonical isomorphism $M(G) \cong M(G^{deg}) \oplus \mathbb{N}^m$, where m is the number of non-distinguished edges in G.*

Proof. The first statement follows from the definition 5.18 and the construction of G^{deg}. To prove the second statment, we first notice that in the construction of G^{deg}, we identify the weights of any two vertices connected by a non-distinguished path, which is equivalent to the equation 5.2.1 given by the non-distinguished edges from the path. It is clear that we have an injection $M(G^{deg}) \to M(G)$. Denote by $\{e_i\}_{i=1}^m$ the set of weights of the non-distinguished edges. First notice that none of the elements in $\{e_i\}$ is involved in the equations 5.2.1. Thus, these elements give the part $\mathbb{N}^m$. Note that $M(G)$ is generated by $M(G^{deg})$ and $\{e_i\}$. It follows that $M(G) \cong M(G^{deg}) \oplus \mathbb{N}^m$. $\square$

Note that we can identify the weights $e_{v,i}$ and e_l with the element in $M(G)$. Denote by $N(G)$ the submonoid of $M(G)$ generated by the weights $e_{v,i}$ and e_l.

htInMonoid **Lemma 5.23.** *The saturation of $N(G)$ in $M(G)^{gp}$ is $M(G)$, namely for any $a \in M(G)$, there exists $b \in N(G)$ an a positive integer m such that $b = m \cdot a$.*

Proof. This follows from the definition of $M(G)$. $\square$

Definition 5.24. The Graph is called admissible if $M(G)$ is a sharp monoid.

Corollary 5.25. *The graph G is admissible if and only if G^{deg} is admissible.*

Proof. This follows directely from proposition 5.22. $\square$

n:AdmGraph **Corollary 5.26.** *If G is admissible, then there is no i-loop in G for any i*

Proof. If there is an i-loop, then the monoid $M(G)$ fails to be sharp, which contradicts the assumption. $\square$

Note that when G is admissible, the monoid $M(G)$ generates a strong converx rational cone $C(M(G))$ in the vector space $M(G)^{gp} \otimes \mathbb{Q}$ (see [Ful93, Page 4]).

lem:IrrEle **Lemma 5.27.** *Consider an irreducible element $a \in M(G)$, where G is admissible. Then there are only two possibilities*

(1) *There is an positive integer n and an i-minimal vertex v, such that $n \cdot a$ is the i-th weight of v.*
(2) *There is an positive integer n and an edge l, such that $n \cdot a$ is the weight of l.*

Proof. Choose the minimal positive integer n such that $n \cdot a \in N(G)$. Assume that $n \cdot a = b + c$, where $b, c \in N(G)$ are non-trivial elements. Since a is an irreducible element of $M(G)$, and G is admissible, then a generates a ray, which is a face of the strong convex rational cone $C(M(G))$. Thus we have positive numbers n_1 and n_2 such that $b = n_1 \cdot a$ and $c = n_2 \cdot a$, which violates the assumption that n is minimal. Thus, the element $n \cdot a$ must satisfy one of the the two possibilities above. $\square$

5.3. **Graph associated to log maps.** Consider a log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S.

alGraphMap **Construction 5.28.** We construct a dual graph G_ξ of ξ as follows:

(1) The vertices of G_ξ is given by the set

$$V(G_\xi) = \{\ v \mid v \text{ is an irreducible component of } C\}.$$

For each $v \in V(G_\xi)$, we associate a k-tuple of weights $(e_{v,i})_{i=1}^k$ such that $e_{v,i}$ is a variable if v degenerate into D_i, and 0 otherwise.

(2) The edges of G_ξ is given by the set

$$E(G_\xi) = \{\ l \mid l \text{ is a node of } C\}.$$

For each $l \in E(G_\xi)$, we associate a k-tuple of non-negative integers $(c_{l,i})_{i=1}^k$ and a variable e_l, such that $c_{l,i}$ is the i-th contact order of the node l as in definition 5.13.

(3) For each $i \in \{1, 2, \cdots, k\}$, we associate an orientation as follows. Let $l \in E(G_\xi)$ be a node joining two irreducible components $v_1, v_2 \in V(G_\xi)$. Then $v_1 \leq_i v_2$ is v_1 is i-lower component and v_2 is the i-upper component of l as in remark 5.14.

Note that the underlying graph of G_ξ is the dual graph of the underlying curve of ξ. Denote by G_ξ^{deg} the contracted graph associated to G_ξ.

Definition 5.29. We call G_ξ the dual graph, and G_ξ^{deg} the degeneracy graph of ξ.

Corollary 5.30. *Let n be the number of non-distinguished nodes in ξ, then we have*

$$M(G_\xi) = M(G_\xi^{deg}) \oplus \mathbb{N}^n,$$

where the generators of $\mathbb{N}^n$ are giving by the weights of the n edges corresponding to the non-distinguished nodes.

Proof. This follows from proposition 5.22. $\square$

Consider a node $l \in E(G_\xi)$. Denote by e'_l the element in $\overline{\mathcal{M}}_S$ which smoothes l, and e_l the weight of l in $M(G_\xi)$. Then consider an irreducible component $v \in V(G_\xi)$. Denote by $e'_{v,i}$ the i-th degeneracy of v in ξ, and $e_{v,i}$ the i-th weight of v in $M(G_\xi)$. We define a correspondance

CanMapChar (5.3.1) $$e_l \mapsto e'_l \quad \text{and} \quad e_{v,i} \mapsto e'_{v,i}$$

op:CanChar **Proposition 5.31.** *Assume that $\mathcal{M}_S$ is fs, then the correspondance (5.3.1) induces a canonical morphism of monoids*

$$\phi : M(G_\xi) \to \overline{\mathcal{M}}_S.$$

Proof. Note that (5.3.1) induces a map $N(G_\xi) \to \overline{\mathcal{M}}_S$. Note that by lemma 5.23, the saturation of $N(G_\xi)$ is $M(G)$. Thus the statement follows from proposition 2.1.

alGraphAdm **Corollary 5.32.** *The graph G_ξ is a k-weighted, oriented and admissible graph.*

Proof. For the weightedness and orientations, we can directly check the definition by using the results from subsection 5.1.

Let us consider the admissibility. For any element $a \in M(G_\xi)$, if a is invertible, then by lemma 5.23, there exists some positive integer m such that $m \cdot a = \sum_i d_i e_i$, where e_i is some weight, and d_i are positive integers. Note that the monoid $\overline{\mathcal{M}}_S$ is sharp. Then the image $\phi(a) = \sum_i d_i \phi(e_i)$ is an invertible element in $\overline{\mathcal{M}}_S$. Thus we have $a = 0$ in $M(G_\xi)$, which proves the statement. □

Remark 5.33. By remark 5.7, it is not hard to see that the monoid $M(G_\xi)$ does not depend on the choice of the global presentation $P \to \overline{\mathcal{M}}_X$.

5.4. **Minimal condition.** We still consider a log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S. With the help of the dual graph of log maps, we are able to describe the minimal condition:

fn:Minimal **Definition 5.34.** The log map ξ is called minimal if the induced canonical map ϕ as in proposition 5.31 is an isomorphism. A family of log maps ξ_T over a scheme T is called minimal if each geometric fiber is minimal.

op:MinOpen **Proposition 5.35.** *Given a family of log maps $\xi = (C \to S, \mathcal{M}_S, f)$ over a scheme S, and assume that $\bar{s} \in S$ is a geometric point such that $\xi_{\bar{s}}$ is minimal. Then there exists an étale neighborhood of $\bar{s}$ with all geometric fibers minimal.*

Proof. By shrinking S, we can assume that S is connected, and we have a lifting $\beta : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$, which gives a chart on the base. We next show that for any $\bar{t} \in S$, the fiber $\xi_{\bar{t}}$ is minimal.

Denote by

$$K_{\bar{t}} = \{\ a \in \overline{\mathcal{M}}_{S,\bar{t}} \mid \beta(a) \text{ is a unit at } \bar{t}\ \}.$$

Note that $K_{\bar{t}}$ is the submonoid of $\overline{\mathcal{M}}_{S,\bar{t}}$. Consider the following composition

$$\overline{\mathcal{M}}_{S,\bar{s}} \to \overline{\mathcal{M}}^{gp}_{S,\bar{s}} \to \overline{\mathcal{M}}^{gp}_{S,\bar{s}}/K^{gp}_{\bar{t}}.$$

Since $\overline{\mathcal{M}}^{gp}_{S,\bar{s}}/K^{gp}_{\bar{t}} \cong \overline{\mathcal{M}}^{gp}_{S,\bar{t}}$, and all the monoids are toric, the above composition induces a map $q : \overline{\mathcal{M}}_{S,\bar{s}} \to \overline{\mathcal{M}}_{S,\bar{t}}$. We construct a new graph from the dual graph $G_{\xi_{\bar{s}}}$ as follows.

(1) For an edge $l \in E(G_{\xi,\bar{s}})$, if $q(e_l) = 0$, then we contract l, and identify the two end vertices of l and the corresponding weights.
(2) For a vertex $v \in V(G_{\xi,\bar{s}})$, if $q(e_{v,i}) = 0$, then we put $e_{v,i} = 0$ in G'.

Other vertices and edges in $G_{\xi_{\bar{s}}}$, and their weights and contact orders remain the same. We denote by G' the resulting graph. Assume that the edge l has two end vertices v_1 and v_2. If l' is another edge joining v_1 and v_2, then we have $c_{l,i} \cdot e_l = c_{l',i} \cdot e_{l'}$, since $\xi_{\bar{s}}$ is minimal. Thus we have $q(e'_l) = 0$, which implies that l' will also be contracted in G'. Consider two vertices $v_1 \leq_i v_2$. If $q(e_{v_1,i}) = 0$ then we have $q(e_{v_2,i}) = 0$. Therefore, the graph G' is k-weighted. And we can check that $\{\leq_i\}$ induces a natural orientation on G'. By our assumption, since all contact order remains the same, the graph G' is in fact the dual graph $G_{\xi_{\bar{t}}}$ of $\xi_{\bar{t}}$.

The construction of G' gives a canonical map of monoids:

generalize (5.4.1) $$q' : \overline{\mathcal{M}}_{S,\bar{s}} \to M(G_{\xi_{\bar{t}}}),$$

which gives the following commutative diagram:

ralizeChar (5.4.2)

$$\begin{tikzcd} & \overline{\mathcal{M}}_{S,\bar{s}} \arrow[dl, "q'"'] \arrow[dr, "q"] & \\ M(G_{\xi_{\bar{t}}}) \arrow[rr] & & \overline{\mathcal{M}}_{S,\bar{t}}, \end{tikzcd}$$

where the bottom map is the canonical map as in proposition 5.31. Note that both q and q' are surjective maps. Now for any $a \in \overline{\mathcal{M}}_{S,\bar{s}}$, we check that $e \in K_{\bar{t}}$, if and only if $q(e) = 0$, if and only if $q'(e) = 0$. Therefore, we can check that the map $M(G_{\xi_{\bar{t}}}) \to \overline{\mathcal{M}}_{S,\bar{t}}$ is an isomorphism. This proves the statement. □

Remark 5.36. Denote by $\mathcal{K}^{min}_{g,n}(X^{log})$ the stack parametrizing minimal log maps to X^{log}, with the fixed genus g, and n-markings. The above proposition 5.35 shows that this is an open substack of the stack $\mathcal{K}^{log}_{g,n}(X^{log})$ of log maps, therefore is an algebraic stack.

5.5. **Log stable maps.**

:MinStable **Definition 5.37.** A log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S is called *log stable* if its underlying map is stable in the usual sense, and $\mathcal{M}_S$ is fs. A family of log maps ξ_T over a scheme T is called log stable if its geometric fiber are log stable. A log stable map is called *minimal log stable* if it satisfies the minimal condition as in definition 5.34.

:DataGamma **Conventions 5.38.** Assume that the DF log structure $\mathcal{M}_X$ on the target is locally free with a global presentation $\mathbb{N}^k \to \overline{\mathcal{M}}_X$, such that the vanishing locus D_i associated to each copy of $\mathbb{N}$ as in remark 2.13 is connected. Then we introduce the convention of discrete data $\Gamma = (\beta, g, N, \mathbf{c})$ where

(1) $\beta \in H^2(X, \mathbb{Z})$ is a curve class in X;
(2) g is a non-negative integer;
(3) N is a finite ordered set which we may take to be $\{1, \cdots, n\}$;
(4) for each $p \in N$ we associate a k-tuple of non-negative integers $(c_{p,i})_{i=1}^k$ such that

ntactOrder (5.5.1)
$$\sum_{p \in N} c_{p,i} = c_1(L_i^\vee) \cap \beta$$

ammaLogMap **Definition 5.39.** Notations and assumptions as in conventions 5.38, the minimal log map $\xi = (C \to S, \mathcal{M}_S, f)$ over a geometric point S is called Γ-*minimal log stable* if

(1) The source curve $(C \to S, \mathcal{M}_S)$ is a log pre-stable curve of genus g with marked points numbered by N.
(2) the curve class $f_*(C) = \beta$.
(3) The contact orders of each marked point $p \in N$ is given by $(c_{p,i})_{i=1}^k$.

A log map ξ' over a scheme T is called Γ-log stable if its geometric fibers are all Γ-minimal log stable. Since we fix all the discrete data, we will omit Γ in the rest of the paper. The arrows between log stable maps is the same as the arrow of minimal log maps in definition 4.3.

Conventions 5.40. We will use $\mathcal{K}^{mst}_{n,g,\beta}(X, \mathcal{M}_X)$ to denote the stack parametrizing minimal log stable maps with genus g, n-markings, and curve class β, and $\mathcal{K}^{log}_{\Gamma}(X, \mathcal{M}_X)$ to denote the stack parametrizing Γ-minimal log stable maps, if $\mathcal{M}_X$ is locally free DF log structure. Note that these are substacks of $\mathcal{K}^{log}$ as in theorem 4.8.

StabLogAlg **Corollary 5.41.** *The stack $\mathcal{K}^{mst}_{n,g,\beta}(X, \mathcal{M}_X)$ is algebraic.*

Proof. This follows from theorem 4.8 and proposition 5.35. □

Remark 5.42. Denote by Λ the set of discrete data Γ as in convention 5.38 with fixed g, n and β. Note that Λ is a finite set. By lemma 5.11, we have the disjoint union

$$\mathcal{K}^{mst}_{n,g,\beta}(X, \mathcal{M}_X) = \bigcup_{\Gamma \in \Lambda} \mathcal{K}^{log}_{\Gamma}(X, \mathcal{M}_X).$$

FiniteAuto

5.6. **Finiteness of automorphisms.**

:LogMapIso **Remark 5.43.** With the target given by a smooth pair, we can simplify definition 4.2 as follows. Consider two log maps $\xi = (C \to S, \mathcal{M}_S, f)$ and $\xi' = (C' \to S, \mathcal{M}'_S, f')$ over a scheme S. An arrow $\xi \to \xi'$ over S is a pair (ρ, θ) as in definition 3.12 such that the following diagram commutes:

$$\begin{tikzcd}
 & & (X, \mathcal{M}_X) \\
(C, \mathcal{M}_C) \arrow[r, "\rho"] \arrow[d] \arrow[urr, bend left, "f"] & (C', \mathcal{M}_{C'}) \arrow[d] \arrow[ur, bend right, "f'"'] & \\
(S, \mathcal{M}_S) \arrow[r, "\theta"] & (S, \mathcal{M}'_S) &
\end{tikzcd}$$

where the square is cartesian of fine log schemes.

Let $\xi = (C \to S, \mathcal{M}_S, f)$ be a log stable map over the geometric point S. We fix a lifting $\overline{\mathcal{M}}_S \to \mathcal{M}_S$, and identify the weights with their images in $\mathcal{M}_S$.

FiniteAuto **Proposition 5.44.** *Notations as above, the set $\mathcal{A}ut_S(\xi)(S)$ is finite.*

Proof. Note that the underlying automorphism of $\underline{f}$ is finite. Fixing an underlying automorphism $(\underline{\rho}, id_S)$, it is enough to show that there are finitely many automorphisms of ξ whose underlying structure are given by (ρ, θ). For simplicity, we assume that $\underline{\rho} = id_C$, and other cases can be proved similarly.

Let (ρ, θ) be an automorphism with the underlying structure given by (id_C, id_S). First we consider a node $l \in E(G_\xi)$. Denote by x and y the local coordinates of l. By a nice choice of coordinates, we can assume that $e_l = \log x + \log y$. Note that we have

$$\rho^\flat(e_l) = \rho(\log x) + \rho(\log y) = \log \rho^*(x) + \log \rho^*(y).$$

Since $\underline{\rho} = id_C$, the weight e_l is fixed by ρ for any l. Same argument shows that the log structure from the marked points are also fixed by ρ.

Now consider an i-minimal vertex $v \in V(G_\xi)$. Locally on the component of v, we have $f^\flat(\delta_i) = e_{v,i} + \log h$, where h is a local invertible section. Note that we have

$$\rho^\flat(e_{v,i} + \log h) = \rho^\flat(e_{v,i}) + \log \rho^*(h) = \rho^\flat(e_{v,i}) + \log h.$$

Since ρ fixes the section $f^\flat(\delta_i)$, the map $\rho^\flat$ also fixes the weight $e_{v,i}$. Thus, the automorphism (ρ, θ) act trivially on all weights from vertices and edges of G_ξ. By lemma 5.23, such (ρ, θ) is finite. □

6. Decomposition of the stack of minimal log stable maps

6.1. **The universal property of minimal log maps.** In this subsection, we fix a log map $\xi = (C \to S, \mathcal{M}_S, f : (C, \mathcal{M}_C) \to (X, \mathcal{M}_X))$ such that the log structure $\mathcal{M}_S$ is fs. Our main result of this section is the following:

UnivMinLog **Theorem 6.1.** *There exists a minimal log map $\xi_{min} = (C \to S, \mathcal{M}_S^{min}, f_{min} : (C, \mathcal{M}_C^{min}) \to (X, \mathcal{M}_X))$ over S, and a map of fs log schemes $\Phi : (S, \mathcal{M}_S) \to (S, \mathcal{M}_S^{min})$, which fits in the following commutative diagram*

rMinLogMap (6.1.1)
$$\begin{tikzcd}
 & & (X, \mathcal{M}_X) \\
(C, \mathcal{M}_C) \arrow[r, "\Phi_C"'] \arrow[d] \arrow[urr, bend left, "f"] & (C, \mathcal{M}_C^{min}) \arrow[d] \arrow[ur, bend left, "f_{min}"'] & \\
(S, \mathcal{M}_S) \arrow[r, "\Phi"] & (S, \mathcal{M}_S^{min}) &
\end{tikzcd}$$

where the square is caterian in the category of fs log schemes. Furthermore, the datum (g, ξ_{min}) is unique up to a unique isomorphism.

Proof. Note that the statement is local on S. Then the theorem follows from lemmas 6.2, 6.3, 6.4, and 6.5 as follows. $\square$

By construction 5.28, for each geometric point $\bar{t} \in S$ we can associate a dual graph $G_{\xi_{\bar{t}}}$ to the fiber $\xi_{\bar{t}}$. It was shown in lemma 5.32 that $G_{\xi_{\bar{t}}}$ is admissible. By proposition 5.31, we have a canonical morphism of monoids $\phi_{\bar{t}} : M(G_{\xi_{\bar{t}}}) \to \overline{\mathcal{M}}_{S,\bar{t}}$.

UniqMinMap **Lemma 6.2.** *Assume that we have a log pre-stable curve $(C \to S, \mathcal{M}_S^{min})$ and a morphism $\Phi : (S, \mathcal{M}_S) \to (S, \mathcal{M}_S^{min})$ such that*

(1) *For each $\bar{s} \in S$, we have a fixed isomorphism $\overline{\mathcal{M}}_{S,\bar{s}}^{min} \cong M(G_{\xi_{\bar{s}}})$.*
(2) *The induced map $\bar{\Phi}_{\bar{s}}^{\flat} : M(G_{\xi_{\bar{s}}}) \cong \overline{\mathcal{M}}_{S,\bar{s}}^{min} \to \overline{\mathcal{M}}_{S,\bar{s}}$ on the level of characteristic is identical to $\phi_{\bar{s}}$.*
(3) *The log pre-stable curve $(C \to S, \mathcal{M}_S)$ is the pull-back of $(C \to S, \mathcal{M}_S^{min})$ via Φ.*

Then we have a unique log map $f_{min} : (C, \mathcal{M}_C^{min}) \to (X, \mathcal{M}_X)$, which fits in diagram 6.1.1. Note that $(C \to S, \mathcal{M}_S^{min}, f_{min})$ forms a minimal log map.

Proof. Since all the underlying maps are fixed, it is enough to construct the map of log structures $f_{min}^{\flat} : f^*(\mathcal{M}_X) \to \mathcal{M}_C^{min}$, which fits in the following commutative diagram

$$\begin{tikzcd}
 & f^*(\mathcal{M}_X) \arrow[dl, "f_{min}^{\flat}"'] \arrow[dr, "f^{\flat}"] & \\
\mathcal{M}_C^{min} \arrow[rr, "\Phi_C^{\flat}"] & & \mathcal{M}_C.
\end{tikzcd}$$

Consider an arbitrary closed point $p \in C$, which lies in an irreducible component corresponding to the vertex $v \in V(G_{\xi_{\bar{s}}})$. Then locally at p, by the description of the log structure on C, we have

ocalDecomp (6.1.2)
$$f^{\flat}(\delta_i) = e_v + \log h,$$

where $e_v \in \mathcal{M}_S$ near $\bar{s}$, and h is a non-zero regular section locally near p. Note that there are two possible cases: if p is a smooth non-marked point, then h is just a locally invertible section; if p is a special point with i-th contact order c_i, then $h = u \cdot \sigma^{c_i}$, where u is a locally invertible section, and σ is a local coordinate function vanishing at p. Note that the underlying map $\underline{\Phi}_C$ is an identity. Thus, to define $f^{\flat}_{min}(\delta_i)$ locally at p, it is enough to find a lifting $\tilde{e}_v \in \mathcal{M}^{min}_S$ of e_v, such that the image of $\tilde{e}_v$ in $\overline{\mathcal{M}}^{min}_S$ is the i-th weight of the vertex v.

We first consider the uniqueness. Assume that we have two lifting $\tilde{e}_v$ and $\tilde{e}'_v$, such that their images in $\overline{\mathcal{M}}^{min}_S$ are given by the i-th weight of v. Then, we have $\tilde{e}_v = \log u + \tilde{e}'_v$ for some locally invertible function u. This implies that

$$\Phi^{\flat}_C(\tilde{e}_v) = \Phi^{\flat}_C(\log u) + \Phi^{\flat}_C(\tilde{e}'_v).$$

Since $\tilde{e}_v$ and $\tilde{e}'_v$ are two lifting of e_v, we have $\Phi^{\flat}_C(\log u) = 1$. Note that the underlying map $\underline{\Phi}_C = id_C$. It follows that $u = 1$. This shows that the lifting is unique.

Now we consider the existence of the lifting. Denote by $\bar{e}_v$ the image of e_v in the characteristic $\overline{\mathcal{M}}_{S,\bar{s}}$. Note that the map of monoids $\bar{\Phi}^{\flat}_{\bar{s}}$ is identical to $\psi_{\bar{s}}$. Then we have a unique element $\bar{e} \in \overline{\mathcal{M}}^{min}_{S,\bar{s}}$, which corresponds to the weight of v in the graph G, and $\bar{\Phi}^{\flat}_{\bar{s}}(\bar{e}) = \bar{e}_v$. Thus, locally we can lift $\bar{e}$ to an element $\tilde{e}_v \in \overline{\mathcal{M}}^{min}_S$ such that $\Phi^{\flat}_{\bar{s}}(\tilde{e}_v) = e_v$. Thus we can define

ocalConstr

$$f^{\flat}_{min}(\delta_i) = \tilde{e}_v + \log h. \tag{6.1.3}$$

It is not hard to see that the above lifting in equation (6.1.3) does not depend on the choice of expression as in equation (6.1.2). Thus the construction in equation 6.1.3 can be glued globally to obtain a unique log map $f^{\flat}_{min}$ as we want. □

We next construct the log prestable curve $(C \to S, \mathcal{M}^{min}_S)$ satisfying the three conditions in the above lemma. Note that the question is local on S. Pick up a point $\bar{s} \in S$. By shrinking S, we can assume that there is a global chart $\beta : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$. Since we have the canonical map $\phi_{\bar{s}} : M(G_{\xi_{\bar{s}}}) \to \overline{\mathcal{M}}_{S,\bar{s}}$. Consider the pre-log structure given by the folloing composition:

$$M(G_{\xi_{\bar{s}}}) \stackrel{\phi_{\bar{s}}}{\to} \overline{\mathcal{M}}_{S,\bar{s}} \stackrel{\beta}{\to} \mathcal{M}_S \stackrel{\exp}{\to} \mathcal{O}_S.$$

Denote by $\mathcal{M}^{min}_S$ the log structure associated to the above pre-log structure. Thus, the construction above gives a global chart $\beta_{min} : M(G) \to \mathcal{M}^{min}_S$ and a natural map $\Phi^{\flat} : \mathcal{M}^{min}_S \to \mathcal{M}_S$.

Note that the construction of $\mathcal{M}^{min}_S$ is depend on the choice of the chart β. Assume that we have another log structure $\mathcal{M}^{min}_1$ and a map $\Phi^{\flat}_1 : \mathcal{M}^{min}_1 \to \mathcal{M}_S$ over S, which is coming from another chart $\beta_1 : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$. Then we have:

inCharUniq

Lemma 6.3. *There is a unique isomorphism of log structures $\mathcal{M}^{min}_1 \to \mathcal{M}^{min}_S$ fitting in the following commutative diagram:*

$$\begin{tikzcd}
\mathcal{M}^{min}_1 \arrow[rr] \arrow[dr, "\Phi^{\flat}_1"'] & & \mathcal{M}^{min}_S \arrow[dl, "\Phi^{\flat}"] \\
& \mathcal{M}_S &
\end{tikzcd}$$

Proof. Consider an irreducible element $a \in M(G)$. By our construction, it image in $\mathcal{M}_S$ via β_1 and β are differ by a unique unit. This proves the lemma. □

mpCurveLog **Lemma 6.4.** *Further shrinking S if necessary, we have a unique dashed arrow which makes the following diagram commute:*

CurveArrow (6.1.4)
$$\begin{tikzcd} \mathcal{M}_S^{min} \arrow[rr, "\Phi^\flat"] & & \mathcal{M}_S \\ & \mathcal{M}_S^{C/S} \arrow[ul, dashed, "\phi_{min}"] \arrow[ur, "\phi"'] & \end{tikzcd}.$$

where ϕ is the structure arrow defining the log pre-stable curve $(C \to S, \mathcal{M}_S)$.

Proof. By futher shrinking S, we can choose a global chart $\mathbb{N}^m \to \mathcal{M}_S^{C/S}$. Let e be a generator of $\mathbb{N}^m$ which corresponds to an edge $l \in V(G_{\xi_{\bar{s}}})$. For convenience, we will identify e with its image in $\mathcal{M}_S^{C/S}$. Consider $\phi(e) \in \mathcal{M}_S$, and its image $\bar{\phi}(e) \in \overline{\mathcal{M}}_S$. Now on the level of characteristic, there is a unique element $\bar{e}' \in \overline{\mathcal{M}}_S^{min}$, which corresponds to the weight of l, such that $\bar{\Phi}^\flat(\bar{e}') = \bar{\phi}(e)$. A similar argument as in the proof of lemma 6.2 shows that there is a unique section $e' \in \mathcal{M}_S^{min}$ such that $\Phi^\flat(e') = \phi(e)$. Then we can define $\phi_{min}(e) = e'$ for all generator e. This gives the map $\phi_{min} : \mathcal{M}_S^{min} \to \mathcal{M}_S$.

Note that our construction depends on a fixed chart $\mathbb{N}^m \to \mathcal{M}_S^{C/S}$. However, a similar argument as in the proof of lemma 6.2 again shows that different choice of the global chart will induces the same map ϕ_{min}. This finishes the proof. ☐

By lemma 5.15, we can further shrink S, and assume that the contact order of the nodes on each geometric fiber is given by the graph $G_{\xi_{\bar{s}}}$. Now we have:

ralization **Lemma 6.5.** *The log structure $\mathcal{M}_S^{min}$ satisfies the condition (1) and (2) in lemma 6.2.*

Proof. The proof of this lemma is identical to the one for proposition 5.35. Indeed, consider diagram 5.4.2, if we replace $\overline{\mathcal{M}}_{S,\bar{s}}$ by $\overline{\mathcal{M}}_{S,\bar{s}}^{min}$, then the bottom arrow gives the isomorphism as in lemma 6.2(1). ☐

uliOverTor **Remark 6.6.** Consider a log stable map $\xi = (C \to S, \mathcal{M}_S, f)$. Then we have a minimal log stable map $\xi_{min} = (C \to S, \mathcal{M}_S^{min}, f_{min})$ and a log map $g : (S, \mathcal{M}_S) \to (S, \mathcal{M}_S^{min})$ satisfy the conditions in theorem 6.1. This induces a unique (up to a unique isomorphism) log map $(S, \mathcal{M}_S) \to \mathcal{K}_{n,g,\beta}^{mst}(X, \mathcal{M}_X)$, such that ξ is obtained by pulling back the universal minimal log stable maps via this map. Here, we view $\mathcal{K}_{n,g,\beta}^{mst}(X, \mathcal{M}_X)$ as a log stack with its log structure given by the universal log structure for the log stable map.

Denote by $\mathcal{T}or_{\mathbb{C}}$ the open substack of $\mathcal{L}og_{\mathbb{C}}$ parametrizing fine and saturated log schemes over $\mathbb{C}$. We refer to [Ols03a] for the construction and properties of $\mathcal{T}or_{\mathbb{C}}$. Then the above argument describes $\mathcal{K}_{n,g,\beta}^{mst}(X, \mathcal{M}_X)$ as a fibered category parametrizing log stable maps over $\mathcal{T}or_{\mathbb{C}}$. Similarly, the stack $\mathcal{K}_{n,g}^{min}(X, \mathcal{M}_X)$ is a fibered category over $\mathcal{T}or_{\mathbb{C}}$, parametrizing log maps with fs log structures.

Remark 6.7. If the log structure $\mathcal{M}_X$ on the target X is trivial, then the stack $\mathcal{K}_\Gamma^{log}(X, \mathcal{M}_X)$ is isomorphic to the stack $\mathcal{K}_{n,g}(X, \beta)$ of usual stable maps with the canonical log structure coming from the universal curve over it.

6.2. **Decomposition of the stack of minimal log stable maps.** Consider a cartesian diagram of fs schemes:

getProduct (6.2.1)

$$\begin{tikzcd} X_0 \arrow[r, "t_{01}"] \arrow[d, "t_{02}"'] & X_1 \arrow[d, "t_{13}"] \\ X_2 \arrow[r, "t_{23}"] & X_3, \end{tikzcd}$$

where the underlying scheme $\underline{X}_i = \underline{X}$ for all i. Denote by $\mathcal{M}_i$ the corresponding log structure of X_i. Consider the following log stack

$$\mathcal{K} := \mathcal{K}^{mst}_{n,g,\beta}(X_1) \times_{\mathcal{K}^{mst}_{n,g,\beta}(X_3)} \mathcal{K}^{mst}_{n,g,\beta}(X_2),$$

where the fiber product is taking in the category of fine log schemes. Given a log stable map $\xi_0 = (C \to S, \mathcal{M}_S, f) \in \mathcal{K}^{mst}_{n,g,\beta}(X_0)$, we have two induced log stabe maps $\xi_1 \in \mathcal{K}^{mst}_{n,g,\beta}(X_1)$ and $\xi_2 \in \mathcal{K}^{mst}_{n,g,\beta}(X_2)$, by composing f with t_{01} and t_{02} respectively. Similarly, from the commutativity of diagram (6.2.1), the images of ξ_1 and ξ_2 in $\mathcal{K}^{mst}_{n,g,\beta}(X_3)$ by composing with t_{13} and t_{23} is identical. Thus, we defined a map of log stacks

$$\mathcal{S}at : \mathcal{K}^{mst}_{n,g,\beta}(X_0) \to \mathcal{K}.$$

ProdDecomp **Proposition 6.8.** *The log stack $\mathcal{K}^{mst}_{n,g,\beta}(X_0)$ is a saturation of $\mathcal{K}$ given by the morphism $\mathcal{S}at$.*

Proof. Consider a log map $g : (S, \mathcal{M}_S) \to \mathcal{K}$, where $\mathcal{M}_S$ is a fs log structure on S. We will show that there is a unique (up to a unique isomorphism) morphism $g' : (S, \mathcal{M}_S) \to \mathcal{K}^{mst}_{n,g,\beta}(X_0)$ such that $g = \mathcal{S}at \circ g'$.

Note that the map g induces a log curve $(C \to S, \mathcal{M}_S)$, and a commutative diagram

$$\begin{tikzcd} (C, \mathcal{M}_C) \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & X_0 \arrow[r] \arrow[d] & X_1 \arrow[d] \\ & X_2 \arrow[r] & X_3, \end{tikzcd}$$

where $\mathcal{M}_C$ is the log structure given by the log curve $(C \to S, \mathcal{M}_S)$. Note that the underlying map $C \to \underline{X}$ is a usual stable map over S, and the dashed arrow is induce by the solid arrows. By theorem 6.1 and remark 6.6, we have a unique (up to a unique isomorphis) arrow $g' : (S, \mathcal{M}_S) \to \mathcal{K}^{mst}_{n,g,\beta}(X_0)$, which is induced by the log stable maps given by the dashed arrow. It is not hard to see that $g = \mathcal{S}at \circ g'$. Now the statement follows from proposition 2.8(2). □

pGeneralDF **Corollary 6.9.** *Consider a DF log pair $X^{log} = (X, \mathcal{M}_X)$ with the locally free presentation as in (2.2.3). Then we have*

$$\mathcal{K}^{mst}_{n,g,\beta}(X^{log}) \cong \mathcal{K}^{mst}_{n,g,\beta}(X^{log}_0) \times_{\mathcal{K}^{mst}_{n,g,\beta}(X^{log}_1)} \mathcal{K}^{mst}_{n,g,\beta}(X^{log}_0),$$

where the fiber product is taking over the category of fs log schemes.

Proof. This follows directly from proposition 6.8, and lemma 2.16. □

compFreeDF **Corollary 6.10.** *Consider a locally free DF log pair $X^{log} = (X, \mathcal{M}_X)$ with the decompostion (2.2.1) of the remark 2.12. Then we have*

$$\mathcal{K}^{mst}_{n,g,\beta}(X^{log}) \cong \mathcal{K}^{mst}_{n,g,\beta}(X_1^{log})_{\mathcal{K}_{n,g}(X,\beta)} \cdots \times_{\mathcal{K}_{n,g}(X,\beta)} \mathcal{K}_{n,g}(X_k^{log}),$$

where we view $\mathcal{K}_{n,g}(X,\beta)$ as log stack with the canonical log structure from its universal curve, and the fiber product is taking over the category of fs log schemes.

Proof. This directly follows from proposition 6.8. ☐

6.3. **Statement of the main theorem.**

thm:Main **Theorem 6.11.** *Given a DF log pair $X^{log} = (X, \mathcal{M}_X)$, the stack $\mathcal{K}^{mst}_{n,g,\beta}(X^{log})$ is a proper Deligne-Mumford stack.*

Proof. By corollary 6.9, 6.10, and proposition 2.8, it is enough to consider the case where $\mathcal{M}_X$ is locally free with a global presentation $\mathbb{N} \to \overline{\mathcal{M}}_X$. Indeed, we will prove that the stack $\mathcal{K}^{mst}_{\Gamma}(X, \mathcal{M}_X)$ is proper and Deligne-Mumford, where $\mathcal{M}_X$ is a DF log structure on X, which is given by a line bundle L with a global section s, such that the vanishing locus of $s^\vee$ is connected.

With the above reduction, the boundedness is proved in section 7, and the weak valuative criterion is proved in section 8. Since the stack has finite diagonal, it was shown in [DEV, Theorem 2.7] that $\mathcal{K}^{mst}_{\Gamma}(X, \mathcal{M}_X)$ admit a finite surjective morphism from a scheme. With this property and the weak valuative criterion, by [GLB00, Proposition 7.12] the stack is proper. Finally the Deligne-Mumford property follows from the stability condition. ☐

oundedness

7. The boundedness theorem for minimal log stable maps

con:Target **Conventions 7.1.** In this and the next section, we fix the log pairs $X^{log} = (X, \mathcal{M}_X)$ as our target of minimal log stable maps, such that there is a global presentation $\mathbb{N} \to \overline{\mathcal{M}}_X$. We use δ to denote the standard generator of $\mathbb{N}$. Since the global presentation locally lifts to a chart, we will identify δ with the corresponding element in $\mathcal{M}_X$, if no confusion would arise. Note that $\mathcal{M}_X$ corresponds to the pair (L, s) consists of a line bundle L, and a global section $s : L \to \mathcal{O}_X$. Let D be the vanishing locus of the dual section $s^\vee$. Without loss of generality, we can assume that D is connected.

oundedness **Theorem 7.2.** *There exists a scheme T of finite type, and a map $T \to \mathcal{K}^{mst}_{\Gamma}(X^{log})$, which exhausts all geometric point of $\mathcal{K}^{mst}_{\Gamma}(X^{log})$. Namely, all geometric point of $\mathcal{K}^{mst}_{\Gamma}(X^{log})$ is contained in the image of T.*

GenericDeg

7.1. **Removing degeneracy from log stable maps.** In this subsection, we fix a log stable map (not necessarily minimal) $\xi = (C \to S, \mathcal{M}_S, f)$ over a connected scheme S, such that there exists a point $\bar{s} \in S$ and a lifting of global chart $\beta : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$. Denote by $\overline{\mathcal{M}}^{deg}_{S,\bar{s}}$ the image of $M(G^{deg}_{\xi_{\bar{s}}})$ in $\overline{\mathcal{M}}_{S,\bar{s}}$. Then we obtain a sub-log structure $\mathcal{M}^{deg}_S$ generated by $\overline{\mathcal{M}}^{deg}_{S,\bar{s}}$ via β. Note that this is a fs log structure on S. In this subsection, we put the following assumption

GenericDeg (7.1.1) The characteristic $\overline{\mathcal{M}}^{deg}_S$ is a constant sheaf of monoids on S.

Remark 7.3. Note that the log structure $\mathcal{M}^{deg}_S$ is fs, and does not depend on the choice of β. By the assumption (7.1.1), we have $\overline{\mathcal{M}}^{deg}_{S,\bar{s}'} = \overline{\mathcal{M}}^{deg}_{S,\bar{s}}$ for any $\bar{s}' \in S$.

ntDegGraph **Lemma 7.4.** *With the assumptions as above, the degeneracy graph $G^{deg}_{\xi_{\bar t}}$ is identical to the degeneracy graph $G^{deg}_{\xi_{\bar s}}$ for any $\bar t \in S$.*

Proof. Note that the elements smoothing the distinguished nodes are in $\overline{\mathcal{M}}^{deg}_S$. Then the statement follows from the assumption (7.1.1). $\square$

Denote by G^{deg}_{ξ} the dual graph of ξ over S. Fixing a global chart β as above, we obtain an induced map $\hat\beta : \overline{\mathcal{M}}^{deg}_S \to \mathcal{M}_C$. Denote by $\hat{\mathcal{M}}_C = \mathcal{M}^{gp}_C/(\overline{\mathcal{M}}^{deg}_S)^{gp}$ the quotient given by the map $\hat\beta$. Consider the following diagram:

ag:QuotDeg (7.1.2)
$$\begin{tikzcd} 0 \arrow[r] & (\overline{\mathcal{M}}^{deg}_S)^{gp} \arrow[r, "\hat\beta^{gp}"] & \mathcal{M}^{gp}_C \arrow[r] & \hat{\mathcal{M}}_C \\ & & f^*(\mathcal{M}_X) \arrow[u] \arrow[ur, "\hat f^\flat"'] & \end{tikzcd}$$

where the map $\hat f^\flat$ is given by the composition $f^*(\mathcal{M}_X) \to \mathcal{M}_C \to \hat{\mathcal{M}}_C$.

DepOnChart **Remark 7.5.** Note that the morphism $\hat f^\flat$ depends on the choice of a lifting $\beta : \overline{\mathcal{M}}_{S,\bar s} \to \mathcal{M}_S$. This will be important when we discuss the valuative criterion.

bDistPoint **Conventions 7.6.** Consider the subcurve C_v of C corresponding to vertex $v \in G^{deg}_\xi$ (or $v \in G_\xi$). Note that C_v is connected. Denote by $\{p_l\}_{l\in\Lambda^{low}_v}$ the set of splitting nodes, joining v with v' for some $v' \leq v$. Let $\{p_l\}_{l\in\Lambda^{up}_v}$ be the set consists of the following special points in C_v:

(1) the set of splitting nodes, joining v with v'' for some $v \leq v''$;
(2) the marked points with non-trivial contact orders.

Denote by c_l the contact order at p_l for $l \in \Lambda^{low}_v \cup \Lambda^{up}_v$. Consider the line bundle

$$L_v = \prod_{l\in\Lambda^{low}_v} \mathcal{O}_{C_v}(c_l\cdot p_l) \otimes \prod_{l\in\Lambda^{up}_v} \mathcal{O}_{C_v}(-c_l\cdot p_l).$$

Note that the line bundle L_v only depends on the graph G^{deg}_ξ.

:LineBdIso **Proposition 7.7.** *Assume that the weight of $v \in G^{deg}_\xi$ is not zero. Then the map $\hat f^\flat$ induces a natural isomorphism of line bundles*

$$\hat f^\flat_v : f^*(L) \to L_v.$$

Proof. We first construct $\hat f^\flat_v$ locally. There are three cases.

Case 1: Consider a closed point p of p_l for $l \in \Lambda^{up}_v$. Locally at p we have

$$f^\flat(\delta) = e_v + c_l \log \sigma_l,$$

where σ_l is the local coordinate of p in C_v defining the marking p_l, and e_v is contained in the image of $\hat\beta$. Thus, we have $\hat f^\flat_v(\delta) = c_l \log\sigma_l$. Then locally near p we define

plitUpNode (7.1.3)
$$\hat f^\flat_v(\delta) = \sigma_l^{c_l},$$

Note that $\sigma_l^{c_l}$ is the local section of L_v at p.

Case 2: Consider a closed point p of the splitting node p_l for $l \in \Lambda_v^{low}$. Assume that p_l joining vertices v' and v such that $v' \leq v$. Locally at p we have

LowNodeLog
$$f^\flat(\delta) = e_{v'} + c_l \log \sigma'_l, \tag{7.1.4}$$

where $e_{v'}$ is in the image of $\hat{\beta}$. By a nice choice of coordinates we have

LowNodeCur
$$c_l \cdot e_l = c_l \log \sigma_l + c_l \log \sigma'_l, \quad \text{in } \mathcal{M}_C \tag{7.1.5}$$

where σ'_l is the local coordinate of p_l in C'_v, and e_l is the element smoothing node, and contained in the image of $\hat{\beta}$. Then we have

$$1 = c_l \log \sigma_l + c_l \log \sigma'_l, \quad \text{in } \hat{\mathcal{M}}_C.$$

This induces

$$\hat{f}^\flat(\delta) = c_l \log \sigma_l = 1 - c_l \log \sigma_l.$$

Then locally at the node p we define

litLowNode
$$\hat{f}^\flat_v(\delta) = (\frac{1}{\sigma_l})^{c_l}. \tag{7.1.6}$$

Note that this is a local generator of L_v at p.

Case 3: Locally at a point p which is not contained in one of the p_l for $l \in \Lambda_v^{up} \cup \Lambda_v^{low}$, we have

$$f^\flat(\delta) = e_v + \log h,$$

where h is an invertible function at p and e_v is contained in the image of $\hat{\beta}$. Then the map $\hat{f}^\flat(\delta) = \log h$ induces

SplitGenPt
$$\hat{f}^\flat_v(\delta_\lambda) = h. \tag{7.1.7}$$

Note that the local construction of $\hat{f}^\flat_v$ is naturally given by $\hat{f}^\flat$, which is a map of sheaves of monoids. Thus these local definitions can be glued to obtain a global map. We also notice that δ lifts to a the local generator of L. Therefore, we construct an isomorphism of line bundles $\hat{f}^\flat_v$ as required. $\square$

7.2. **Finiteness of the discrete data.** The splitting technique introduced in last section allows us to extract the discrete data from non-degenerated maps rather than log maps. Now we will use this idea to obtain the following:

FinDisData

Proposition 7.8. *The following set is finite:*

$$\{G \mid G \text{ is the dual graph of some } \xi \in \mathcal{K}_\Gamma^{mst}(X^{log})(\mathbb{C})\}.$$

Proof. Step 1: Bound the choices of underlying dual graph. Denote by $\mathcal{K}_{n,g}(X,\beta)$ the Kontsevich moduli space of stable maps, with n-marked points, genus g, and curve class β in X. Note that we have a morphism

$$\mathcal{K}_\Gamma^{mst}(X^{log}) \to \mathcal{K}_{n,g}(X,\beta),$$

by removing all log structures. Let $U \to \mathcal{K}_{n,g}(X,\beta)$ be an affine étale chart. Consider the following cartesien diagram:

$$\begin{tikzcd} \mathcal{K}_U \arrow[r] \arrow[d] & \mathcal{K}_\Gamma^{mst}(X,D) \arrow[d] \\ U \arrow[r] & \mathcal{K}_{n,g}(X,\beta). \end{tikzcd}$$

Since the stack $\mathcal{K}_{n,g}(X,\beta)$ is of finite type, it is enough to prove that the dual graph corresponds to the geometric point of $\mathcal{K}_U$ is finite. Denote by $C_U \to U$ the universal curve and $\underline{f}_U : C_U \to X$ the universal map over U.

Since U is of finite type, it is covered by finite strata such that the family of curves over each stratum has fixed dual graph. For our purpose, we can put the reduced scheme structure on each stratum. So we can pick up a such stratum S such that its universal curve with dual graph G, and reduced scheme structure. Denote by $\underline{f} : C \to X$ the unidersal map over S.

Step 2: Bound the choices of distinguished nodes and orientations. Since G is a finite graph, the choices of distinguished nodes are finite. We first fix a such choice on G, and denote by G^{deg} the graph obtained by contract the non-distinguished edges in G. These are also finite, for the finiteness of G^{deg}. So we fix an orientation on G^{deg} such that:

(1) If C_v does not degenerate to D, then v is a minimal vertex, for any $v \in G^{deg}$
(2) For any two adjacent vertices $v, v' \in V(G^{deg})$, only one of the two conditions $v \leq v'$ and $v' \leq v$ is satisfied.
(3) No loops are allowed.

In this way, we obtain an orientation on G.

Step 3: Bound the choices of contact orders. Since we have fixed the orientation and distinguished nodes, for any $v \in G$, we can use the notations $\{p_l\}_{l\in\Lambda_v^{low}}$ and $\{p_l\}_{l\in\Lambda_v^{up}}$ for the two sets of distinguished points on the subcurve C_v as in conventions 7.6. Denote by c_l the possible contact order at the distinguished point p_l. Since the dual graph of the underlying curve is fixed, the multidegree of $\underline{f}^*(L)$ on C_v is fixed for any $v \in V(G)$. By propersition 7.7, we have

qu:CompDeg
$$\deg \underline{f}^*(L)|_{C_v} = \sum_{l\in\Lambda_v^{low}} c_l - \sum_{l'\in\Lambda_v^{up}} c_{l'}. \tag{7.2.1}$$

First, consider a maximal vertex $v \in V(G)$. Then the set $\{p_l\}l \in \Lambda_v^{up}$ is given by the discrete data Γ. Since the contact orders are all positive, the choices of c_l for $l \in \Lambda_v^{low}$ is finite by equation (7.2.1).

Consider an arbitrary vertex $v' \in V(G^{deg})$. We assume that for any adjacent vertex v of v' such that $v' \leq v$, the number of choice of the contact orders along the splitting nodes joining v' and v is finite. Then by taking into account all contact orders from adjacent vertices and those from marked points of C, a similar argument shows that the possible choices of contact order c_l for $l \in \Lambda_{v'}^{low}$ are finite.

Finally consider an edge l joining two vertices v_1 and v_2 in G, such that v_2 degenerates to D_2, but v_1 does not. Then the contact order along l is determined by the underlying map. Since G is a finite graph, this proves that the choice of contact orders on G is finite.

This finishes the proof of the proposition. ☐

7.3. **Proof of theorem 7.2.** Consider the family of usual stable maps $\underline{f} : C \to X$ over S as in step 1 of the above proof. Fix a possible admissible graph G_0 with $\underline{G_0} = G$ the dual graph of C. We use the notations as in step 3 of the above proof, and assume that the equation (7.2.1) holds for the fixed graph G_0. Since the stack $\mathcal{K}_{n,g}(X,\beta)$ is of finite type, to prove theorem 7.2, it is enough to prove the following:

austStrata **Proposition 7.9.** *Notations and assumptions as above, there exists a scheme T of finite type over S, and a family of minimal log stable maps ξ over T, which satisfies the following conditions: for any minimal log map ξ' over $\bar{s}$, with dual graph given by G_0, and underlying map $\underline{\xi}'$ given by the pull-back of $\underline{f}$ via $\bar{s} \to S$, there exists a lifting $\bar{s} \to T$, such that ξ' is isomorphic to the pull-back $\xi_{\bar{s}}$.*

Proof. By shrinking S, we can assume that S is affine, and the canonical log structure $\mathcal{M}_S^{C/S}$ on S coming from the family $C \to S$ has a global chart $\mathbb{N}^n \cong \overline{\mathcal{M}}_{S,\bar{s}}^{C/S} \to \mathcal{M}_S$ for some geometric point $\bar{s} \in S$. Consider the pre-log structure $M(G_0) \to \mathcal{O}_S$, given by $e \mapsto 0$ for any non-trivial element $e \in M(G_0)$. Denote by $\mathcal{M}_S$ the new log structure associated to the pre-log structure. Note that there is a natural map $\mathbb{N}^n \to M(G_0)$ giving by the corresponding nodes. This induces a natural map $\mathcal{M}_S^{C/S} \to \mathcal{M}_S$, hence a log pre-stable curve $\zeta = (C \to S, \mathcal{M}_S)$ over S. Note that any minimal log map ξ' over $\bar{s} \in S$ as in the statement has the source log curve isomorphic to $\zeta_{\bar{s}}$.

Denote by $\mathcal{M}_C$ the log structure on C according to the log pre-stable curve ζ. Note that over C we have another log structure $\underline{f}^*(\mathcal{M}_X)$. Since the dual graph G_0 is fixed, we have a morphism of locally constant sheaves on C:

$$\bar{f}^\flat : \underline{f}^*(\overline{\mathcal{M}}_X) \to \overline{\mathcal{M}}_C,$$

which is locally described as in subsection 5.1. To define a log map $f : (C, \mathcal{M}_C) \to X^{log}$, it is enough to define a map of log structures $f^\flat : \underline{f}^*(\mathcal{M}_X) \to \mathcal{M}_C$ fitting in the following commutative diagram:

iftCharMap (7.3.1)

$$\begin{tikzcd}
 & \underline{f}^*(\mathcal{M}_X) \arrow[r, dashed, "f^\flat"] \arrow[d, "p_1"] & \mathcal{M}_C \arrow[d, "p_2"] \\
\mathbb{N} \arrow[r] & \underline{f}^*(\overline{\mathcal{M}}_X) \arrow[r, "\bar{f}^\flat"] & \overline{\mathcal{M}}_C
\end{tikzcd},$$

where the two vertical arrows are the canonical projection, and the arrow $\mathbb{N} \to \underline{f}^*(\overline{\mathcal{M}}_X)$ is the pull-back of the global presentation. Note that the arrow $\bar{f}^\flat$ is an injection. Denote by δ_X and δ_C the image of δ in $\underline{f}^*(\overline{\mathcal{M}}_X)$ and $\overline{\mathcal{M}}_C$ respectively. The inverse image $p_1^{-1}(\delta_X)$ and $p_2^{-1}(\delta_C)$ form two $\mathcal{O}_C^*$-torsors. Thus to have a dashed arrow fitting in diagram (7.3.1), it is equivalent to have a global section of the presheaf $\mathcal{I}som_C(p_1^{-1}(\delta_X), p_2^{-1}(\delta_C))$ of isomorphisms of two torsors over C. Note that the torsor $p_1^{-1}(\delta_X)$ corresponds to the line bundle $\underline{f}^*L$. Denote by L_C the corresponding line bundle of $p_2^{-1}(\delta_C)$. Then we have

$$\mathcal{I}som_C(p_1^{-1}(\delta_X), p_2^{-1}(\delta_C)) \cong \mathcal{I}som_C(\underline{f}^*L, L_C) \cong \mathcal{I}som_C(\underline{f}^*L \otimes L_C, \mathcal{O}_C).$$

Denote by I the above presheaves. It is well-known that line bundles are parametrized by the algebraic stack $\mathcal{B}G_m$. Thus, I is a sheaf represented by an separated algebraic space of finite type. Let $\pi : C \to S$ be the projection. By [Ols06, theorem 1.5], there is an algebraic space $\pi_* I$ locally of finite type over S, which for any $Y \to S$ associates the groupoid of isomorphisms $(\underline{f}^*L \otimes L_C)_Y^{-1} \to \mathcal{O}_{C_Y}$. We have the following lemma for the boundedness of $\pi_* I$.

Lemma 7.10. *The algebraic space $\pi_* I$ is of finite type over S.*

Proof. Note that the two line bundles L_C and f^*L have the same degree when restrict to each irreducible component over $\bar{s} \in S$. Since S is affine, by [FP97, Proposition 1], there is a unique closed subscheme $T \subset S$ which represents the condition that $f^*L \otimes L_{C_T}^{-1}$ is a trivial line bundle. We fix an isomorphism $\phi : f^*L \otimes L_{C_T}^{-1} \cong \mathcal{O}_{C_T}$. Consider the scheme $U := T \times \mathbb{G}_m$ with the pull-back isomorphism ϕ_U over U. We define a new isomorphism $\phi'_U : f^*L \otimes L_{C_T}^{-1} \cong \mathcal{O}_{C_T}$ given by the scalar multiplication of $\mathbb{G}_m$ on ϕ_U. It is not hard to see that the family ϕ'_U induces a map $U \to \pi_*I$, which exhausts the geometric points of π_*I. Then the statement follows from the fact that U is of finite type over S. □

By pulling back via $\pi_*I \to S$, we have a family of log pre-stable curves $\zeta_{\pi_*I} = (C_I \to \pi_*I, \mathcal{M}_{\pi_*I})$, a usual stable map $\underline{f}_{\pi_*I} : C_I \to X$, and a morphism of sheaves of monoids $f^\flat_{\pi_*I} : \underline{f}^*_{\pi_*I}\mathcal{M}_X \to \mathcal{M}_{C_I}$, where $\mathcal{M}_{C_I}$ is the log structure on C_I given by the log curve ζ_{π_*I}.

Lemma 7.11. *The points $\bar{t} \in \pi_*I$, whose fiber $f^\flat_{\pi_*I,\bar{t}}$ gives a morphism of log structures, forms a closed subset of π_*I.*

Proof. The condition $f^\flat_{\pi_*I}$ is a morphism of log structures is equivalent to have the following commutative diagram:

g:BdLogMap (7.3.2)
$$\begin{tikzcd} \underline{f}^*_{\pi_*I}\mathcal{M}_X \arrow[rr, "f^\flat_{\pi_*I}"] \arrow[dr, "\exp_X"'] & & \mathcal{M}_{C_I} \arrow[dl, "\exp_C"] \\ & \mathcal{O}_{C_I} & \end{tikzcd} ,$$

where the two arrows $\exp_X$ and $\exp_C$ are the structure maps of corresponding log structure. Locally on C_I, we choose a generator $\delta \in \underline{f}^*_{\pi_*I}\mathcal{M}_X$, then the commutativity of the diagram is equivalent to
$$\exp_X(\delta) = \exp_C \circ f^\flat_{\pi_*I}(\delta),$$
which is clearly a closed condition. Let $V \subset C_I$ be the closed subscheme represents the commutativity of diagram (7.3.2) over C_I, and V^c the complement of V in C_I. Denote by W the image of V^c in π_*I via the projection $C_I \to \pi_*I$. Since the family of curves is flat, the image W is open in π_*I. Thus, the complement W^c of W is closed in π_*I. This proves the lemma. □

Now we take $T = W^c$ as in the above proof with the reduced scheme structure. Then W^c is a closed subscheme of π_*I, note that by pulling back families over π_*I, we have a family of minimal log maps ξ over T. According to our construction, the family ξ over T satisfies the lifting property as in proposition 7.9. □

Theorem 7.2 follows from the above arguments. □

:Valuative

8. The weak valuative criterion for minimal log stable maps

We keep using the notations for target as in 7.1. Let R be a discrete valuation ring, and K the fraction field of R. Denote by π the uniformizer of R, and $S =$Spec$R$. Let s and η be the closed and generic point of S respectively. Let R' be another discrete valuation ring, and π_0 be its uniformizer. Denote by s' and η' the closed and generic point of $S' =$SpecR' respectively.

thm:Val **Theorem 8.1.** *With the notations above, given ξ_η a minimal log stable map over η. Possibly after an base change given by an injection $R \hookrightarrow R'$ of DVR, which induce a finite extension* *of fraction fields, we have an extension of minimal log stable maps given by the following cartesian diagram:*

$$\begin{array}{ccc} \xi_{\eta'} & \longrightarrow & \xi_{S'} \\ \big\downarrow & & \big\downarrow \\ \eta' & \longrightarrow & S', \end{array}$$

where $\xi_{\eta'}$ is the pull-back of ξ_η via $\eta' \to \eta$, and $\xi_{S'}$ is a minimal log stable map over S'. Furthermore, the extension $\xi_{S'}$ is unique up to a unique isomorphism and its formation commutes with further injections of discrete valuation rings.

Proof. Since the underlying structure of minimal log stable maps are the usual stable maps, possibly after base change, we have a usual stable map $\underline{f} : C \to S$ over the new base, such that its restriction to the generic fiber is given by the pull-back of $\underline{\xi}_\eta$. For simplicity, we still use S to denote the new base. The theorem follows from subsections 8.4 and 8.2. $\square$

8.1. **Local analysis of the extended underlying map.** We first consider the case ξ_η is log stable map (not necessarily minimal). We still use $\underline{f} : C \to X$ to denote the extended underlying stable map over S. Possibly after a base change, we fix a chart $\beta_\eta : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$. Denote by G_η the dual graph of ξ_η. If a node of C is smoothed out over η, then we call it a *special node*, otherwise we call it a *generic node*.

Consider a point $p \in C_{\bar{s}}$, and pick up an étale neighborhood $p \in U$. Consider over the generic point η. By shrinking U, we can assume that over U_η we have

erGenPoint (8.1.1) $$f^\flat(\delta) = \beta_\eta(e_v) + \log u_p,$$

where $e_v \in \overline{\mathcal{M}}_\eta$ is the degeneracy of some vertex $v \in G_\eta$, and $u_p \in \mathcal{O}_{U_\eta}$. Note that U is normal, and u_p extend to a rational function on U. Denote by ν_π the evaluation of the divisor given by the uniformizer π. Let $n_p = \nu_\pi(u_p)$, then we have the following result.

Underlying **Lemma 8.2.** *Assume that p is not a generic node, and U is connected, and does not contain a generic node. Further shrinking U, the point p has to satisfy one of the following possibilities:*

1. *If p is a smooth point, and there is a neighborhood of p, which contains only non-distinguished points over η, then we have $u_p = \pi^{n_p} \cdot h_p$, where $h_p \in \mathcal{O}_U^*$.*
2. *If p is specialized from a marked point with contact order c over η, then $u_p = \pi^{n_p} \cdot x^c \cdot h_p$, where $h_p \in \mathcal{O}_U^*$, and the section containing p is given by the vanishing of $x \in \mathcal{O}_U$.*
3. *If p is a special node, then $u_p = \pi^{n_p} \cdot x^c \cdot h_p$, where $h_p \in \mathcal{O}_U^*$, the section $x \in \mathcal{O}_U$ is the local coordinate of one component of p, and c is a non-negative integer.*

Note that if in (3) we have $c = 0$, then this compatible with the case described in (1).

Proof. Since $n_p = \nu_\pi(u_p)$, and u_p is well-defined over the generic fiber, we have $u_p = \pi^{n_p} \cdot h'_p$ for some $h'_p \in \mathcal{O}_U$. If there is a neighborhood of p, which contains only non-distinguished points over η, then $h'_p \in \mathcal{O}^*_{U_\eta}$. Since $\nu_\pi(h'_p) = 0$, then $h'_p \in \mathcal{O}^*_U$. This proves (1).

For (2), we have $\nu_x(h'_p) = c$, where ν_x is the evaluation map given by the divisor corresponding to the vanishing of x. Then we have $h'_p = x^c \cdot h_p$, such that the restriction $h_p|_{U_\eta}$ is invertible. The same argument as for (1) shows that $h_p \in \mathcal{O}^*_U$.

Consider the case where p is a special node. Denote by x and y the local coordinates of the two components meeting at p. By a nice choice of the coordinates, we can assume that $x \cdot y = \pi^n$ for some positive integer n. Without loss of generality, we can assume that $\nu_y(h'_p) = 0$ and $\nu_x(h'_p) = c$ for some non-negative integer c. Thus, same as for (2), we have $h'_p = x^c \cdot h_p$ for some $h_p \in \mathcal{O}_U^*$. This proves (3). □

SpecialDeg **Remark 8.3.** Note that the rational section u_p hence the integer n_p in (8.1.1) are depend on the choice of the chart β_η. We call the integer n_p *the special degeneracy of p under the chart β_η*. Let Z be an irreducible component over $\bar{s}$ that contains p, then it is not hard to see that generic points on Z also have n_p as the special degeneracy under β_η. Thus, we call n_p *the special degeneracy of Z under β_η*.

ecialOrder **Remark 8.4.** Consider a node p joining two irreducible components Z_1 and Z_2 over $\bar{s}$. First we assume that p is a special node. Let x and y be the local coordinates on Z_1 and Z_2 near p respectively, such that $x \cdot y = \pi^n$. By lemma 8.2(3), we can assume that $u_p = \pi^{n_p} \cdot x^c \cdot h_p$. Thus, we can check that the special degeneracy of Z_1 is n_p, and the special degeneracy of Z_2 is $n_p + c \cdot n$. In this case, we denote $Z_1 \leq Z_2$. Note that if $c = 0$, we have both $Z_1 \leq Z_2$, and $Z_2 \leq Z_1$.

Consider the case where p is a generic node. We take the normalization of C along all the generic node. Then we obtain a set of usual pre-stable curves $\{C_v\}_{v \in V(G_\eta)}$ over S. If $Z_1 \subset C_{v_1}$ and $Z_2 \subset C_{v_2}$, and $v_1 \leq v_2$, then we define $Z_1 \leq Z_2$. In this way, we define an orientation on the dual grpah $G_{\bar{s}}$ of the underlying curve $C_{\bar{s}}$.

SpecialDeg **Lemma 8.5.** *Using the notations as above, consider another chart $\beta'_\eta : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$. The two special degeneracy of p given by β_η and β'_η are the same, if and only if $\beta'_\eta(e_v) = \log u + \beta_\eta(e_v)$ for some $u \in R^*$.*

Proof. Same as equation (8.1.1), we have

$$f^\flat(\delta) = \beta'_\eta(e_v) + \log u'_p.$$

Then

$$u \cdot u'_p = u_p.$$

Now the statement follows from the proof of lemma 8.2(1). □

erTangency **Lemma 8.6.** *With the notations as above, the integer c as in (2) and (3) of lemma 8.2 does not depend on the choice of chart β_η. Therefore the orientation on $G_{\bar{s}}$ defined in remark 8.4 does not depend on the choice of chart of β_η.*

Proof. Consider another chart $\beta_\eta : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$, and

$$f^\flat(\delta) = \beta'_\eta(e_v) + \log u'_p,$$

for some $u'_p \in \mathcal{O}^*_{U_\eta}$. Then we have

$$\beta'_\eta(e_v) = \beta_\eta(e_v) + \log a,$$

for some element $a \in K$. Comparing with (8.1.1), we have

$$u_p = a \cdot u'_p.$$

Since c is given by the evaluation ν_x, it is not hard to see that the statement holds. □

Now we turn to the case where p is a generic node. Denote by p_η the corresponding node over the generic point. Again we have the rational section u_p over U as in equation (8.1.1). By shrinking U, we can choose two regular sections x and y on U_η, which correspond to the coordinates of the two components meeting at p_η. By a nice choice of coordinates, we can assume that

GenNodeDVR
$$\beta_\eta(e_l) = \log x + \log y, \quad \text{in} \quad U_\eta. \tag{8.1.2}$$

Without loss of generality, we can assume that $u_p = x^c$, where c is the contact order at p_η. Then u_p vanishes along the component with coordinate y.

Let $l \in E(G)$ be the edge corresponding to the generic node p, and assume that l joins two vertices v_1 and v_2. By taking the normalization of U along the generic node given by l, we obtain two sub-schemes U_1 and U_2 of U. By shrinking U, we can assume that $U_i \subset C_{v_i}$ for $i = 1, 2$. We sill use x and y to denote restriction of x and y to U_1 and U_2 respectively, and p_i the pre-image of p in U_i for $i = 1, 2$. Then x and y can be viewed as a rational function on U_1 and U_2 respectively. Let Σ_1 and Σ_2 be the two sections in U_1 and U_2 respectively, which are coming from the splitting node corresponding to l. Let σ_i be the regular functions on U_i, whose vanishing gives the section Σ_i for $i = 1, 2$.

derGenNode **Lemma 8.7.** *With the notations as above, we have*

(1) $x = \pi^{n_1} \cdot \sigma_1 \cdot h_1$, *where* $n_1 = \nu_\pi(x)$, *and* $h_1 \in \mathcal{O}^*_{U_1}$.
(2) $y = \pi^{n_2} \cdot \sigma_2 \cdot h_2$, *where* $n_1 = \nu_\pi(y)$, *and* $h_2 \in \mathcal{O}^*_{U_2}$.

Proof. The proof of this is similar to that for lemma 8.2. □

derGenNode **Remark 8.8.** Note that σ_1 and σ_2 forms the coordinates at p. By a nice choice of coordinates, we can assume that h_1 and h_2 in the lemma 8.7 is 1. Thus, we have $u_p = \pi^{c \cdot n_1} \cdot \sigma_1^c$.

s:ExistExt

8.2. **Existence of the extension.** Consider the minimal log stable map ξ_η as in theorem 8.1. Denote by $C(\overline{\mathcal{M}}_\eta)$ the convex rational polyhedral cone of $\overline{\mathcal{M}}_\eta$ in $\overline{\mathcal{M}}_\eta^{gp} \otimes_{\mathbb{Z}} \mathbb{Q}$. Since $\overline{\mathcal{M}}_\eta$ is sharp, the cone $C(\overline{\mathcal{M}}_\eta)$ is strongly convex.

Lemma 8.9. *There is a lattice point* $\tilde{v} \in \overline{\mathcal{M}}_\eta^{gp}$ *such that* $(u, \tilde{v}) > 0$ *for any non-zero element* $u \in C(\overline{\mathcal{M}}_\eta)$, *where* $(\cdot, \cdot)$ *is the standard dot product in the Euclidean space* $\overline{\mathcal{M}}_\eta^{gp} \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. This follows from [Ful93, Section 1.2(iv)]. □

We fix a lattice point $\tilde{v}$ satisfies the condition in the above lemma. The set

$$\{(u, \tilde{v}) \mid u \in C(\overline{\mathcal{M}}_\eta)\} \subset \mathbb{Q}$$

forms a rank one free monoid $\mathbb{N}$. Thus, we have a map of monoid $l_{\tilde{v}} : \overline{\mathcal{M}}_\eta \to \mathbb{N}$. Consider the log structure $\mathcal{M}'_\eta$, associated to the pre-log structure $\mathbb{N} \to K, \quad e \mapsto 0$ over η. We fix a chart $\beta_\eta : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$ and $\beta'_\eta : \overline{\mathcal{M}}'_\eta \cong \mathbb{N} \to \mathcal{M}'_\eta$. Then we have a morphism of log structures $\mathcal{M}_\eta \to \mathcal{M}'_\eta$ given by

$$\beta_\eta(e) \mapsto \beta'_\eta \circ l_{\tilde{v}}(e).$$

Denote by ξ'_η the log stable map obtained by pulling back ξ_η via the map $(\eta, \mathcal{M}_\eta) \to (\eta, \mathcal{M}'_\eta)$. By Theorem 6.1, it is enough to construct a log stable map (not necessarily minimal) ξ', such that its generic fiber is given by ξ'_η as above.

tiveSpeDeg **Lemma 8.10.** *Using the notations as above, there exists a chart* $\beta'_\eta : \overline{\mathcal{M}}'_\eta \to \mathcal{M}'_\eta$, *such that no components of* C *over* $\bar{s}$ *have negative special degeneracy under* β'_η *as in remark 8.3.*

Proof. We fix a chart β'_η as above. Consider an irreducible component Z over the closed point $\bar{s}$. Let $p \in Z$ be a smooth non-distinguished point $p \in Z$. Consider the nearby points of p over η. By lemma 8.2, we have

hangeChart

$$f^\flat(\delta) = \beta'_\eta(e) + \log \pi^n \cdot u, \tag{8.2.1}$$

where u is a locally invertible section near p, and $e \in \overline{\mathcal{M}}'_\eta$. If $n \geq 0$, then there is nothing to prove. Consider the case $n < 0$. Since the number of irreducible components over $\bar{s}$ is finite, we can assume that n is the minimal special degeneracy of Z under β'_η. Consider the new chart given by

fnNewChart

$$\beta''_\eta : \overline{\mathcal{M}}'_\eta \to \mathcal{M}_\eta, \quad e \mapsto \beta'_\eta(e) - n \cdot \log \pi. \tag{8.2.2}$$

It is not hard to check that the equation (8.2.1) becomes

$$f^\flat(\delta) = \beta''_\eta(e) + \log u.$$

Since n is minimal, we can check that no irreducible components of C over $\bar{s}$ have negative special degeneracy under β''_η. $\square$

We fix a chart $\beta'_\eta : \overline{\mathcal{M}}'_\eta \to \mathcal{M}_\eta$, which satisfies the condition in lemma 8.10. Consider the log structure $\mathcal{M}'_S$ associated to the following pre-log structure on S:

$$\mathbb{N}^2 \to R, \quad e_\eta \mapsto 0, \text{ and } e_s \mapsto \pi,$$

where e_η and e_s form the basis of $\mathbb{N}^2$. Now we identify $\mathcal{M}'_{S,\eta}$ with $\mathcal{M}'_\eta$, and the element e_η corresponds to the chart $\beta'_\eta : \overline{\mathcal{M}}'_\eta \to \mathcal{M}_\eta$.

Lemma 8.11. *With the notations as above, there is a unique morphism of log structures $\mathcal{M}_S^{C/S} \to \mathcal{M}'_S$, whose restriction to the generic point η is identical to the morphism of log structures $\mathcal{M}_\eta^{C_\eta/\eta} \to \mathcal{M}'_\eta$ given by ξ'_η.*

Proof. Possibly after a base change, we can choose a global chart $\beta^{C/S} : \overline{\mathcal{M}}_{S,\bar{s}}^{C/S} \to \mathcal{M}_S^{C/S}$. Denote by G the dual graph of $C_{\bar{s}}$, and $\{e_l\}_{l \in E(G)}$ the set of generators of $\overline{\mathcal{M}}_S$, such that $\beta^{C/S}(e_l)$ is an element in $\mathcal{M}_S$ smoothing the node corresponding to l in the closed fiber. Assume that l is smoothed out over η, then $\exp \circ \beta^{C/S}(e_l) = \pi^n \cdot h$, where n is an positive integer, and h is an invertible element in R. Thus, we define

$$e_l \mapsto n \cdot e_s + \log h.$$

If the node corresponding to l persists over η, then we have

$$e_l \mapsto n_\eta \cdot e_\eta + \log \pi^{n_s} + \log h, \quad \text{over } \eta$$

where n_η and n_s are non-negative integers, and h is an invertible element in R. Note that this can be done by choosing a sufficiently large n in equation (8.2.2). Thus, we define

$$e_l \mapsto n_\eta \cdot e_\eta + n_s \cdot e_s + \log h.$$

This induces a map $\mathcal{M}_S^{C/S} \to \mathcal{M}'_S$, whose restriction to the generic point coincides with $\mathcal{M}_\eta^{C_\eta/\eta} \to \mathcal{M}'_\eta$. The uniqueness follows from our construction. $\square$

Note that the map $\mathcal{M}_S^{C/S} \to \mathcal{M}'_S$ in the above lemma gives a log pre-stable curves $(C \to S, \mathcal{M}'_S)$, whose restriction to η is given by the log-prestable curve $(C_\eta \to \eta, \mathcal{M}'_\eta)$ of ξ'_η.

LimitExist **Proposition 8.12.** *There is a unique log stable map ξ' over $(S, \mathcal{M}'_S)$ with the log curve $(C \to S, \mathcal{M}'_S)$, whose restriction to η is identical to ξ'_η.*

Proof. It is enough to define the morphism of log structures $f^\flat : \underline{f}^*\mathcal{M}_X \to \mathcal{M}'_C$, where $\mathcal{M}'_C$ is the log structure on C corresponding to the log curve $(C \to S, \mathcal{M}'_S)$. Pick up a point $p \in C$ over $\bar{s}$, and an étale neighborhood U of p. By shrinking U, we can assume that over the generic point, we have

$$f^\flat_\eta(\delta) = n \cdot e_\eta + \log u_p, \quad \text{in } U_\eta,$$

where $u_p \in \mathcal{O}_{U_\eta}$.

We first assume that p is not a generic node. By lemma 8.2, further shrinking U if necessary, the section u_p extend to U of the following form:

$$u_p = \pi^{n_1} \cdot h',$$

where $n_1 = \nu_\pi(u_p)$, and $h' \in \mathcal{O}_U$. Note that the lemma 8.10 implies that the integer n_1 is non-negative. Thus, the only possible way to define $f^\flat$ near p is given by

$$f^\flat(\delta) = n \cdot e_\eta + n_1 \cdot e_s + \log h'.$$

Next we consider the case p is a generic node. With the notations in remark 8.8, we have

$$u_p = \pi^{c \cdot n} \cdot \sigma_1^c.$$

Thus, we define

$$f^\flat(\delta) = n \cdot e_\eta + n_1 \cdot e_s + c \cdot \log \sigma_1.$$

It is not hard to check that such local construction can be glued together to obtain a global map $f^\flat$ as we want. □

:SpecGraph 8.3. **Specializing the dual graph.** Consider the dual graph G of the underlying curve $C_{\bar{s}}$. For each edge $l \in E(G)$, if l corresponds to a special node, then we can associate to l an non-negative integer c given by lemma 8.2; if l corresponds to a generic node, then we associate to l the contact order given by ξ_η. Note that remark 8.4 gives an orientation on G, which is compatible with the contact orders defined on each edge. Thus, we obtain a weighted orientated graph. We still use G to denote this graph with the discrete data.

p:UniGraph **Proposition 8.13.** *Consider a minimal log stable map ξ over S, which is an extension of ξ_η. Then the dual graph $G_{\xi_{\bar{s}}}$ is identical to the graph G with the orientation and contact orders as above.*

Proof. First note that the underlying graph $G_{\xi_{\bar{s}}}$ is given by the dual graph of $C_{\bar{s}}$. Hence the two graphs $G_{\xi_{\bar{s}}}$ and G has the same underlying graph. It is enough to check that the two graph has the same contact orders and orientations. We denote the underlying graph to be $\underline{G}$. Consider an edge $l \in E(\underline{G})$. If l corresponds to a generic node, then by lemma 5.11, the orientation and contact order of l is unquely determined by the generic fiber ξ_η. Hence the two graphs $G_{\xi_{\bar{s}}}$ and G has the same orientation and contact order along l.

Next, consider the case where l corresponds to a special node p. Assume that the contact order of ξ at p is c. Since the log structure is compatible with the underlying structure, the contact order c coincides with the integer c in lemma 8.2(3), and the orientation of l in $G_{\xi_{\bar{s}}}$ is given by the one described in remark 8.4. This implies that the two graphs $G_{\xi_{\bar{s}}}$ and G has the same orientation and contact order along l.

This finishes the proof of the statement. □

Corollary 8.14. *The graph G is admissible.*

Proof. This follows from the existence of the extension of ξ_η and the above proposition. $\square$

Consider a minimal log stable map $\xi = (C \to S, \mathcal{M}_S, f)$, which is an extension of ξ_η over S. Consider the natural map $q^{gen} : M(G)^{gp} \cong \overline{\mathcal{M}}^{gp}_{S,\bar{s}} \to \overline{\mathcal{M}}^{gp}_{\eta}$. This is a surjection. Denote by K_{sp} the kernel of q^{gen}. Then we have the following exact sequence:

$$0 \to K_{sp} \to M(G)^{gp} \to \overline{\mathcal{M}}^{gp}_{\eta} \to 0.$$

Note that all groups involved in the above sequence are free abelian groups. Thus, we have the natural decomposition:

DecompChar
$$M(G) = K_{sp} \oplus \overline{\mathcal{M}}^{gp}_{\eta}. \tag{8.3.1}$$

Denote by $q^{sp} : M(G)^{gp} \to K$ the natural projection. Then for any element $e \in M(G)^{gp}$, we have $e = q^{gen}(e) + q^{sp}(e)$.

Remark 8.15. Note that the dual graph G_{ξ_η} can be obtained by contracting edges in G, which correspond to the special nodes. The map q^{gen} is obtained in a similar way as for equation (5.4.1). This shows that the two maps q^{gen}, q^{sp}, and the decompositin (8.3.1) are independent of the choice of the minimal log stable map ξ over S.

Possibly after a base change, we fix a global chart $\beta : M(G) \cong \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$. By [Ols03a, 3.5(i)], the group K_{sp} is generated by elements in $M(G)$, whose image in R is not zero. Consider the composition

$$\bar{\beta} := \nu_\pi \circ \exp \circ \beta^{gp} : K_{sp} \to \mathbb{Z},$$

where ν_π is the evaluation of of the fraction field K.

SpecialEle **Lemma 8.16.** *The map $\bar{\beta}$ only depends on the base S and the generic fiber.*

Proof. Consider an irreducible element $e \in M(G)$, whose image in R via β is $a \neq 0$. Without loss of generality, we can assume that $a = \pi^n$. By lemma 5.27, there is a minimal positive integer n', such that ne is the weight of some minimal vertex or the weight of some special node l in G. In the first case, the the minimal vertex is specialized from a non-degenerate component over η. Hence, lemma 8.2(1) implies that the degeneracy $n \cdot n'$ is uniquely determined by the generic fiber. If ne is the weight of a special node, then this is determined by the generic fiber, for the underlying map is stable in the usual sense. This proves the statement.$\square$

Consider the map β'_η given by the composition

$$\overline{\mathcal{M}}_\eta \longrightarrow \overline{\mathcal{M}}^{gp}_\eta \longrightarrow \overline{\mathcal{M}}^{gp}_{S,\bar{s}} \xrightarrow{\beta^{gp}} \mathcal{M}^{gp}_S.$$

It is not hard to see that this gives a chart for $\mathcal{M}_\eta$.

cializable **Definition 8.17.** A chart $\beta'_\eta : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$ is called specializable, if it is coming from a global chart $\beta : \overline{\mathcal{M}}_{S,\bar{s}} \to \mathcal{M}_S$ as above.

For any element $e \in \overline{\mathcal{M}}_{S,\bar{s}}$, consider the decomposition

$$e = q^{sp}(e) + q^{gen} = e^{sp} + e^{gen}.$$

Then we have

:SpeSpeDeg
$$\beta(e)_\eta = \beta^{gp}(e^{sp})_\eta + \beta'_\eta(e^{gen}), \quad \text{in} \quad \mathcal{M}_\eta. \tag{8.3.2}$$

Note that $\exp \circ \beta^{gp}(e^{sp}) \in K$, hence is an invertible element in $\mathcal{M}_\eta$.

:SpeSpeDeg **Lemma 8.18.** *Let $Z \subset C_{\bar{s}}$ be an irreducible component, which corresponds to $v \in V(G)$. Then the special degeneracy under the specializable chart β'_η is given by $\bar{\beta} \circ q^{sp}(e_v)$, where $e_v \in M(G)$ is the weight of v.*

Proof. This follows directly from the definition of specializable chart, and equation (8.3.2). ☐

niqueLimit

8.4. **Uniqueness of the extension.** Assume that we have two minimal log stable extensions $\xi_1 = (C \to S, \mathcal{M}_1, f_1)$ and $\xi_2 = (C \to S, \mathcal{M}_2, f_2)$ of ξ_η. After a base change, we can assume that we have two global chart

u:TwoChart (8.4.1)
$$\beta_1 : M(G) \to \mathcal{M}_1 \quad \text{and} \quad \beta_2 : M(G) \to \mathcal{M}_2.$$

for ξ_1 and ξ_2 respectively.

m:ChartDif **Lemma 8.19.** *For any element $e \in M(G)$, we have a unique element $u \in R^*$ such that*

$$\beta_1(a)_\eta = u \cdot \beta_2(a)_\eta \quad \text{in } \mathcal{M}_\eta.$$

Thus, we have a canonical isomorphism of log structures $\mathcal{M}_1 \cong \mathcal{M}_2$.

Proof. We only need to consider the irreducible elements of $M(G)$. Let a be an irreducible element of $M(G)$. By lemma 5.27, We have a minimal positive integer n such that $n \cdot a \in N(G)$ is either the weight of some edge, or the weight of some minimal vertex.

Consider the case that $n \cdot a$ is the weight of some edge l. We identify the element $e_l \in \mathcal{M}_S^{C/S}$ smoothing l with its image in $\mathcal{M}_1$ or $\mathcal{M}_2$, then we have

$$n \cdot \beta_1(a) + \log u_1 = e_l \ \text{ in } \ \mathcal{M}_1, \quad \text{and} \quad n \cdot \beta_2(a) + \log u_2 = e_l \ \text{ in } \ \mathcal{M}_2,$$

where $u_1, u_2 \in R^*$. By restricting to the generic point η, we have

$$e_{l,\eta} = n \cdot \beta_1(a)_\eta + \log u_1 = n \cdot \beta_2(a)_\eta + \log u_2 \ \text{ in } \ \mathcal{M}_\eta.$$

This implies that

$$\beta_1(a)_\eta = \log u + \beta_2(a)_\eta \quad \text{in } \mathcal{M}_\eta,$$

where $u \in R^*$ such that $u^n = u_2/u_1$. Since we fixed the isomorphism $\xi_{1,\eta} \cong \xi_{2,\eta} \cong \xi_\eta$, the element u is unique.

Next, we consider the case where $n \cdot a$ is the weight of some minimal vertex $v' \in V(G)$, and assume that v' is specialized from $v \in V(G_\eta)$. Denote by $a^{sp} = q^{sp}(a)$ and $a^{gen} = q^{gen}(a)$. By lemma 8.18, without loss of generality, we can assume that

$$\exp \circ \beta_1^{gp}(a^{sp}) = \exp \circ \beta_1^{gp}(a^{sp}) = \pi^{n'},$$

where $n' = \bar{\beta}(a^{sp})$. By equation (8.3.2), we have

$$n \cdot \beta_1(a)_\eta = n \cdot n' \cdot \log \pi + n \cdot \beta'_{1,\eta}(a^{gen})$$

and

$$n \cdot \beta_2(a)_\eta = n \cdot n' \cdot \log \pi + n \cdot \beta'_{2,\eta}(a^{gen}),$$

where $\beta_{i,\eta}$ is the specializable chart induced by β_i for $i = 1, 2$. By lemma 8.18 and 8.5, there is an element $u \in R^*$, such that

$$\beta'_{2,\eta}(a^{gen}) = \log u + \beta'_{1,\eta}(a^{gen}).$$

Note that such u is unique, since we fixed the isomorphism $\xi_{1,\eta} \cong \xi_{2,\eta}$. This proves the statement. ☐

imitUnique **Proposition 8.20.** *Possibly after an extension, the isomorphism $\xi_{1,\eta} \cong \xi_{2,\eta}$ extends uniquely to an isomorphism of ξ_1 and ξ_2.*

Proof. For simplicity, we assume that $\xi_{1,\eta} = \xi_{2,\eta} = \xi_\eta$ We fix two global chart β_1 and β_2 as in equation (8.4.1). Denote by $\beta_{i,\eta} : \overline{\mathcal{M}}_\eta \to \mathcal{M}_\eta$ the specializable chart induced by β_i for $i = 1, 2$. By lemma 8.19, we can identigy $\mathcal{M}_1$ and $\mathcal{M}_2$. Thus the two chart $\beta_{1,\eta}$ and $\beta_{2,\eta}$ are identical.

We first show that the following diagram commutes:

$$\begin{tikzcd} & \mathcal{M}_S^{C/S} \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\ \mathcal{M}_1 \arrow[rr, equal] & & \mathcal{M}_2, \end{tikzcd}$$

where ψ_i is the structure map defining the corresponding log curve of ξ_i. Since we put the standard log structure along non-distinguished nodes, we only need to consider the distinguished nodes over the closed point. Consider a distinguished node p over the closed point. Let $e_p \in \mathcal{M}_S$ be a section smoothing p. Then we have

$$\psi_1(e_p) = \psi_2(e_p) + \log u.$$

where u is a unit in R. Since $\xi_{1,\eta} = \xi_{2,\eta} = \xi_\eta$, by restricting the above equation to the generic point η, we obtian $u = 1$. This proves the commutativity. Thus, we can identify the two log curves of ξ_1 and ξ_2.

It remains to show that the two morphisms of log structures $f^\flat_{\xi_i}$ for $i = 1, 2$ are identical. Pick up a point $p \in C$ over $\bar{s}$, then we need to prove that locally at p we have

CompLogMap (8.4.2)
$$f^\flat_{\xi_1}(\delta) = f^\flat_{\xi_2}(\delta).$$

Since the two log stable maps ξ_1 and ξ_2 are minimal, then locally at p we have

$$\bar{f}^\flat_{\xi_1}(\delta) = \bar{f}^\flat_{\xi_2}(\delta) \text{ in } \overline{\mathcal{M}}_S.$$

Thus, locally at p, there exists an invertible function u such that

$$f^\flat_{\xi_1}(\delta) = f^\flat_{\xi_2}(\delta) + \log u.$$

Since $\xi_{1,\eta} = \xi_{2,\eta}$, by restricting to the generic fiber, we obtained that $u = 1$. This proves the equality (8.4.2) at p. Therefore, the statement in the proposition holds. □

References

Artin [Art74] M. Artin, *Versal deformations and algebraic stacks*, Invent. Math **27** (1974), 165–189.

AV [AV02] Dan Abramovich and Angelo Vistoli, *Compactifying the space of stable maps*, J. Amer. Math. Soc. **15** (2002), no. 1, 27–75.

EHKV [DEV] A. Kresch D. Edidin, B. Hassett and A. Vistoli, *Brauer groups and quotient stacks*, American Journal of Mathematics.

FP [FP97] W. Fulton and R. Pandharipande, *Notes on stable maps and quantum cohomology*, arXiv:alg-geom/9608011v2 (1997).

Toric [Ful93] William Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, 1993.

EGAIII(1) [GD61] Alexander Grothendieck and Jean Dieudonné, *éléments de géométrie algébrique iii. étude cohomologique des faisceaux cohérents. i*, Inst. Hautes Études Sci. Publ. Math., 1961.

LMB [GLB00] G.Laumon and L.Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik undihrer Grenzgebiete 39, Springer-Verlag, 2000.

KKato [Kat89] Kazuya Kato, *Logarithmic structures of fontaine-illusie, in:algebraic analysis, geometry, and number theory(Baltimore,MD,1988)*, Johns Hopkins Univ.Press, Baltimore, MD (1989), 191–224.

FKato [Kat00] Fumiharu Kato, *Log smooth deformation and moduli of log smooth curves*, Internat. J. Math. **48 (1996)** (2000), no. 2, 215–232.

Kim [Kim] Bumsig Kim, *Logarithmic stable maps*, arXiv:0807.3611V1.

Ogus [Ogu06] Arthur Ogus, *Lectures on logarithmic algebraic geometry*, TeXed notes (2006).

LogStack [Ols03a] Martin C. Olsson, *Logarithmic geometry and algebraic stacks*, Ann.Sient.Éc.Norm.Sup.(4) **36** (2003), no. 5, 747–791.

LogSS [Ols03b] ______, *Universal log structures on semi-stable varieties*, Tohoku Math. J.(2) **55** (2003), no. 3, 397–438.

LogCot [Ols05] ______, *The logarithmic cotangent complex*, Math.Ann. **333** (2005), no. 4, 859–931.

HomStack [Ols06] ______, *Hom-stack and restriction of scalars*, Duke Math. J. **134** (2006), no. 1.

LogCurve [Ols07] ______, *(log) teisted curves*, Compos.Math. **142** (2007), no. 2, 476–494.

Mochizuki [S.M95] S.Mochizuki, *The geometry of the compactification of the hurwitz scheme*, Publ. Res. Inst. Math. Sci. **31** (1995), no. 3, 355–441.

Dan Abramovich, Department of Mathematics, Box 1917, Brown University, Providence, RI, 02912, U.S.A

Qile Chen, Department of Mathematics, Box 1917, Brown University, Providence, RI, 02912, U.S.A
E-mail address: qile.chen@gmail.com