Addendum to Logarithmic geometry and moduli Lectures at the Sophus Lie Center

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June 16-17, 2014

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Addendum to Logarithmic geometry and more

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Prestable curves

A prestable n-marked curve C/S is a flat, proper morphism with connected reduced fibers of dimension 1, along with disjoint sections $s_i: S \to C$ for i = 1, ..., n in the smooth locus of C/S. We require all fibers have at most nodes as singularities.

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Definition

A prestable curve C/S is *stable* if for every geometric fiber the automorphism group Aut (C_0, p_1, \ldots, p_n) is finite.

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Proposition

All three definitions coincide

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Moduli of stable curves

Theorem (Deligne-Mumford-Knudsen)

Stable curves form a proper, smooth Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}$ over \mathbb{Z} with projective coarse moduli space. The universal curve is $\overline{\mathcal{M}}_{g,n+1}$.



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The reason is that moduli functors dance around the problem instead of facing it directly - the problem of automorphisms.

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With the forgetful functor $(X \rightarrow S) \mapsto S$ this is a category fibered in groupoids.

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If $f_i : C_1|_{U_i} \to C_2|_{U_i}$ are maps between families over S which agree on $U_i \cap U_j$, then there is a glued map f.

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If C_i are families over U_i and $\phi_{i,j}$ are isomorphisms of $C_i|_{U_i \cap U_j}$ with $C_j|_{U_i \cap U_j}$ which are compatible on triple intersections then there is a glued family $C \to S$.

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A map $Z \to \mathcal{M}$ is smooth if it provides geometric objects of \mathcal{M} with versal deformation spaces

A stack is algebraic essentially if it admits a smooth map from a scheme. For instance, $\overline{\mathcal{M}}_g$ is an algebraic stack since it has a smooth map from the Hilbert scheme of 3-canonically embedded stable curve.

Artin's criteria

Michael Artin listed criteria for a moduli problem to be an algebraic stack

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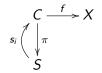
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Often the crucial criterion is the existence of versal deformation spaces.

Stable maps

A stable map is a diagram



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Stable maps

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$$C \xrightarrow{f} X$$

$$s_i \left(\begin{array}{c} \downarrow \pi \\ \downarrow \pi \\ S \end{array} \right)$$

where

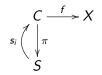
• $(C/S, s_i)$ is a prestable curve,

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where

- $(C/S, s_i)$ is a prestable curve, and
- in fibers $Aut(C_s \rightarrow X, s_i)$ is finite.

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Gromov-Witten theory

We want to count curves on X of class $\beta \in H_2(X, \mathbb{Z})$ meeting cycles $\Gamma_1, \ldots, \Gamma_n$ corresponding to cohomology classes γ_i . For instance: lines through p_1, p_2 .

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There are evaluation maps

$$\begin{array}{rccc} M & \stackrel{e_i}{\to} & X \\ (C/S, p_i) & \mapsto & f(p_i) \end{array}$$

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and one defines the Gromov-Witten invariants

$$\langle \gamma_1 \cdots \gamma_n \rangle_{g,\beta}^{X} = \int_{[M]^{\mathrm{vir}}} e_1^* \gamma_1 \cdots e_n^* \gamma_n.$$

Gromov–Witten theory (continued)

The mysterious part is $[M]^{vir}$. This is there to make this a homological and deformation invariant.

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This is akin to the fact that the number of lines through p_1 , p_2 , namely the intersection number of the locus of lines through p_1 with the locus of lines through p_2 , is 1, whether or not $p_1 = p_2$.

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In order to define this one uses a *perfect obstruction theory*. In this case it is given by $R^{\bullet}\pi_*f^*T_X$, represented by a 2-term complex on S.