Addendum to Logarithmic Geometry and Moduli

Tropical Geometry and Moduli Spaces
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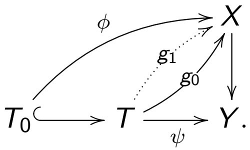
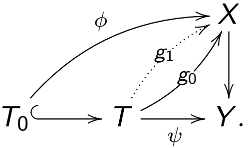
Differentials

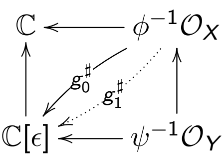
Say $T_0 = \operatorname{Spec} k$ and $T = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, and consider a morphism $X \to Y$.
We contemplate the following diagram:

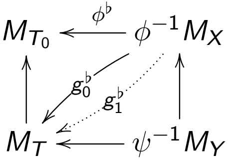


Differentials (continued)

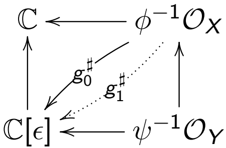


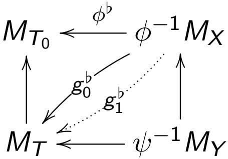
This translates to a diagram of groups and a diagram of monoids





Differentials (continued)





The difference $g_1^\sharp - g_0^\sharp$ is a derivation $\phi^{-1}\mathcal{O}_X \xrightarrow{d} \epsilon\mathbb{C} \simeq \mathbb{C}$
It comes from the sequence

$$0 \to J \to \mathcal{O}_{\underline{T}} \to \mathcal{O}_{\underline{T}_0} \to 0.$$

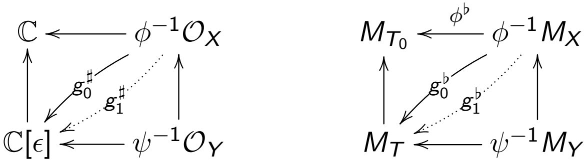
The multiplicative analogue

$$1 \to (1+J) \to \mathcal{O}_{\underline{T}}^\times \to \mathcal{O}_{\underline{T}_0}^\times \to 1$$

means, if all the logarithmic structures are integral,

$$1 \to (1+J) \to M_T \to M_{T_0} \to 1.$$

Differentials (continued)



$$1 \to (1+J) \to M_T \to M_{T_0} \to 1$$

means that we can take the "difference"

$$g_1^\flat(m) = (1 + D(m)) + g_0^\flat(m).$$

Namely $D(m) =$ "$g_1^\flat(m) - g_0^\flat(m)$" $\in J$.

Key properties:

- $D(m_1 + m_2) = D(m_1) + D(m_2)$
- $D|_{\psi^{-1}M_Y} = 0$
- $\alpha(m) \cdot D(m) = d(\alpha(m))$,

in other words,

$$D(m) \quad = \quad d \log\left(\alpha(m)\right),$$

which justifies the name of the theory.

Definition

A logarithmic derivation:

$$\begin{aligned} d : \mathcal{O} &\quad\to\quad J; \\ D : M &\quad\to\quad J \end{aligned}$$

satisfying the above.

Logarithmic derivations

Definition

A logarithmic derivation:

$$\begin{aligned} d : \mathcal{O} &\to J; \\ D : M &\to J \end{aligned}$$

satisfying the above.

The universal derivation:

$$d : \mathcal{O} \to \Omega^1_{\underline{X}/\underline{Y}} = \mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}/\text{relations}$$

The universal logarithmic derivation takes values in

$$\Omega^1_{X/Y} = \left(\Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O} \otimes_{\mathbb{Z}} M^{\mathrm{gp}})\right)/\text{relations}$$

Prestable curves

A *prestable n-marked curve C/S* is a flat, proper morphism with connected reduced fibers of dimension 1, along with disjoint sections $s_i : S \to C$ for $i = 1, \ldots, n$ in the smooth locus of C/S. We require all fibers have at most nodes as singularities.
We denote by p_i the images of s_i.

Stable curves

Definition

A prestable curve C/S is *stable* if for every geometric fiber the automorphism group $\mathrm{Aut}(C_0, p_1, \ldots, p_n)$ is finite.

Definition

A prestable curve C/S is *stable* if for every irreducible component C' of the normalization C^ν of a geometric fiber

- If $C' \simeq \mathbb{P}^1$ then C' contains at least 3 special points.
- If $g(C') = 1$ then C' contains at least 1 special point.

Definition

A prestable curve C/S is *stable* if $\omega_{C/S}(\sum p_i)$ is π-ample.

Proposition

All three definitions coincide

Moduli of stable curves

Theorem (Deligne–Mumford–Knudsen)

Stable curves form a proper, smooth Deligne–Mumford stack $\overline{\mathcal{M}}_{g,n}$ *over* $\mathbb{Z}$ *with projective coarse moduli space. The universal curve is* $\overline{\mathcal{M}}_{g,n+1}$.

Stacks

What is a moduli problem?

We all learned about "representable functors"

These work sometimes, but often replaced by "coarse moduli spaces", a compromise

The reason is that moduli functors dance around the problem instead of facing it directly - the problem of automorphisms.

Moduli as Categories

The object of interest are families $X \to S$.
First and foremost: Families can be pulled back.
So they form a category $\mathcal{M}$, arrows being cartesian diagrams

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ S_1 & \longrightarrow & S_2 \end{array}$$

With the forgetful functor $(X \to S) \mapsto S$ this is a category fibered in groupoids.

Moduli as Stacks

Second, both maps between families in $\mathcal{M}$ and the families themselves can be glued:

If $f_i : C_1|_{U_i} \to C_2|_{U_i}$ are maps between families over S which agree on $U_i \cap U_j$, then there is a glued map f.

If C_i are families over U_i and $\phi_{i,j}$ are isomorphisms of $C_i|_{U_i \cap U_j}$ with $C_j|_{U_i \cap U_j}$ which are compatible on triple intersections then there is a glued family $C \to S$.

This makes the category $\mathcal{M}$ into a stack.

Algebraic stacks

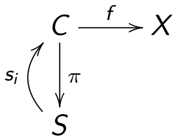
What makes a stack algebraic is that it is approximated by a scheme.
A scheme Z defines a stack which is the category $\mathfrak{Sch}/Z$.
A map $Z \to \mathcal{M}$ is smooth if it provides geometric objects of $\mathcal{M}$ with versal deformation spaces
A stack is algebraic essentially if it admits a smooth map from a scheme.
For instance, $\overline{\mathcal{M}}_g$ is an algebraic stack since it has a smooth map from the Hilbert scheme of 3-canonically embedded stable curve.

Artin's criteria

Michael Artin listed criteria for a moduli problem to be an algebraic stack
Most of them often apply by general nonsense (though I have met surprises!)
Often the crucial criterion is the existence of versal deformation spaces.

Stable maps

A **stable map** is a diagram



where

- $(C/S, s_i)$ is a prestable curve, and
- in fibers $\text{Aut}(C_s \to X, s_i)$ is finite.

Gromov–Witten theory

We want to count curves on X of class $\beta \in H_2(X, \mathbb{Z})$ meeting cycles $\Gamma_1, \ldots, \Gamma_n$ corresponding to cohomology classes γ_i. For instance: lines through p_1, p_2.
Kontsevich's method: the moduli of stable maps $M := \overline{\mathcal{M}}_{g,n,\beta}(X)$ is a Deligne–Mumford stack with projective coarse moduli space.
There are evaluation maps

$$\begin{array}{ccc} M & \stackrel{e_i}{\to} & X \\ (C/S, p_i) & \mapsto & f(p_i) \end{array}$$

and one defines the *Gromov–Witten invariants*

$$\langle \gamma_1 \cdots \gamma_n \rangle^X_{g,\beta} = \int_{[M]^{\text{vir}}} e_1^* \gamma_1 \cdots e_n^* \gamma_n.$$

Gromov–Witten theory (continued)

The mysterious part is $[M]^{\text{vir}}$. This is there to make this a homological and deformation invariant.

This is akin to the fact that the number of lines through p_1, p_2, namely the intersection number of the locus of lines through p_1 with the locus of lines through p_2, is 1, whether or not $p_1 = p_2$.

In order to define this one uses a *perfect obstruction theory*. In this case it is given by $R^\bullet \pi_* f^* T_X$, represented by a 2-term complex on S.