Logarithmic geometry and moduli

Lectures at the Sophus Lie Center

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Heros:

- Olsson
- Chen, Gillam, Huang, Satriano, Sun
- Gross - Siebert
- $\overline{\mathcal{M}}_g$

Moduli of curves

$\mathcal{M}_g$ - a quasiprojective variety.

> *Working with a non-complete moduli space is like keeping change in a pocket with holes*
>
> Angelo Vistoli

Deligne–Mumford

- $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ - moduli of *stable* curves, a modular compactification.
- allow only nodes as singularities
- What's so great about nodes?
- One answer: from the point of view of logarithmic geometry, these are the *logarithmically smooth* curves.

Heros:

- Olsson; Chen, Gillam, Huang, Satriano, Sun; Gross - Siebert
- $\overline{\mathcal{M}}_g$
- K. Kato

Logarithmic structures

Definition

A *pre logarithmic structure* is

$$X = (\underline{X}, M \stackrel{\alpha}{\to} \mathcal{O}_{\underline{X}}) \qquad \text{or just} \quad (\underline{X}, M)$$

such that

- $\underline{X}$ is a scheme - the *underlying scheme*
- M is a sheaf of monoids on X, and
- α is a monoid homomorphism, where the monoid structure on $\mathcal{O}_{\underline{X}}$ is the multiplicative structure.

Definition

It is a *logarithmic structure* if $\alpha : \alpha^{-1}\mathcal{O}_{\underline{X}}^* \to \mathcal{O}_{\underline{X}}^*$ is an isomorphism.

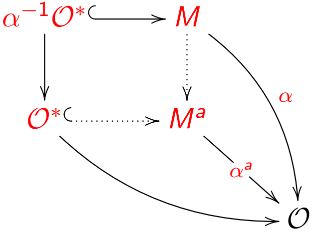
"Trivial" examples

Examples

- $(\underline{X}, \mathcal{O}^*_{\underline{X}} \hookrightarrow \mathcal{O}_{\underline{X}})$, the trivial *logarithmic* structure.
  We sometimes write just $\underline{X}$ for this structure.
- $(\underline{X}, \mathcal{O}_{\underline{X}} \stackrel{\sim}{\to} \mathcal{O}_{\underline{X}})$, looks as easy but surprisingly not interesting, and
- $(\underline{X}, \mathbb{N} \stackrel{\alpha}{\to} \mathcal{O}_{\underline{X}})$, where α is determined by an arbitrary choice of $\alpha(1)$.
  This one is important but only pre-logarithmic.

The associated logarithmic structure

You can always fix a pre-logarithmic structure:



Key examples

Example (Divisorial logarithmic structure)

Let $\underline{X}, D \subset \underline{X}$ be a variety with a divisor. We define $M_D \hookrightarrow \mathcal{O}_{\underline{X}}$:

$$M_D(U) = \left\{ f \in \mathcal{O}_{\underline{X}}(U) \;\middle|\; f_{U \smallsetminus D} \in \mathcal{O}_{\underline{X}}^{\times}(U \smallsetminus D) \right\}.$$

This is particularly important for normal crossings divisors and toric divisors - these will be logarithmically smooth structures.

Example (Standard logarithmic point)

Let k be a field,

$$\begin{array}{ccc} \mathbb{N} \oplus k^\times & \to & k \\ (n, z) & \mapsto & z \cdot 0^n \end{array}$$

defined by sending $0 \mapsto 1$ and $n \mapsto 0$ otherwise.

Works with P a monoid with $P^\times = 0$, giving the *P-logarithmic point*. This is what you get when you restrict the structure on an affine toric variety associated to P to the maximal ideal generated by $\{p \neq 0\}$.

Morphisms

A morphism of (pre)-logarithmic schemes $f : X \to Y$ consists of

- $\underline{f} : \underline{X} \to \underline{Y}$
- A homomorphism $f^\flat$ making the following diagram commutative:

$$\begin{array}{ccc} M_X & \xleftarrow{f^\flat} & \underline{f}^{-1} M_Y \\ \downarrow{\scriptstyle \alpha_X} & & \downarrow{\scriptstyle \alpha_Y} \\ \mathcal{O}_{\underline{X}} & \xleftarrow{\underline{f}^\sharp} & \underline{f}^{-1}\mathcal{O}_{\underline{Y}} \end{array}$$

Definition (**Inverse image**)

Given $\underline{f} : \underline{X} \to \underline{Y}$ and $Y = (\underline{Y}, M_Y)$ define the *pre-logarithmic inverse image* by composing

$$\underline{f}^{-1}M_Y \to \underline{f}^{-1}\mathcal{O}_{\underline{Y}} \stackrel{\underline{f}^\sharp}{\to} \mathcal{O}_{\underline{X}}$$

and then the *logarithmic inverse image* is defined as

$$\underline{f}^* M_Y = (\underline{f}^{-1}M_Y)^a.$$

This is the universal logarithmic structure on $\underline{X}$ with commutative

$$\begin{array}{ccc} (\underline{X}, \underline{f}^* M_Y) & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \underline{X} & \longrightarrow & \underline{Y} \end{array}$$

$X \to Y$ is **strict** if $M_X = \underline{f}^* M_Y$.

Definition (Fibered products)

The fibered product $X \times_Z Y$ is defined as follows:

- $\underline{X \times_Z Y} = \underline{X} \times_{\underline{Z}} \underline{Y}$
- If N is the pushout of

$$\begin{array}{ccc} & \pi_Z^{-1} M_Z & \\ \swarrow & & \searrow \\ \pi_X^{-1} M_X & & \pi_Y^{-1} M_Y \end{array}$$

then the log structure on $X \times_Z Y$ is defined by N^a.

Definition (The spectrum of a Monoid algebras)

Let P be a monoid, R a ring. We obtain a monoid algebra $R[P]$ and a scheme $\underline{X} = \operatorname{Spec} R[P]$. There is an evident monoid homomorphism $P \to R[P]$ inducing sheaf homomorphism $P_X \to \mathcal{O}_{\underline{X}}$, a pre-logarithmic structure, giving rise to a logarithmic structure

$$(P_X)^a \to \mathcal{O}_{\underline{X}}.$$

This is a basic example. It deserves a notation:

$$X = \operatorname{Spec}(P \to R[P]).$$

The most basic example is $X_0 = \operatorname{Spec}(P \to \mathbb{Z}[P])$.

The morphism $\underline{f} : \operatorname{Spec}(R[P]) \to \operatorname{Spec}(\mathbb{Z}[P])$ gives

$$X = \underline{X} \times_{\underline{X_0}} X_0.$$

Charts

A *chart* for X is given by a monoid P and a sheaf homomorphism $P_X \to \mathcal{O}_{\underline{X}}$ to which X is associated.
This is the same as a strict morphism $X \to \mathrm{Spec}(P \to \mathbb{Z}[P])$
Given a morphism of logarithmic schemes $f : X \to Y$, a chart for f is a triple

$$(P_X \to M_X, Q_Y \to M_Y, Q \to P)$$

such that

- $P_X \to M_X$ and $Q_Y \to M_Y$ are charts for M_X and M_Y, and
- the diagram

$$\begin{array}{ccc} Q_X & \longrightarrow & f^{-1}M_Y \\ \downarrow & & \downarrow \\ P_X & \longrightarrow & M_X \end{array}$$

  is commutative.

Types of logarithmic structures

- We say that $(\underline{X}, M_X)$ is *coherent* if *étale locally* at every point there is a *finitely generated* monoid P and a local chart $P_X \to \mathcal{O}_{\underline{X}}$ for X.

- A monoid P is *integral* if $P \to P^{\mathrm{gp}}$ is injective.
- It is *saturated* if integral and whenever $p \in P^{\mathrm{gp}}$ and $m \cdot p \in P$ for some integrer $m > 0$ then $p \in P$. I.e., not like $\{0, 2, 3, \ldots\}$.

- We say that a logarithmic structure is *fine* if it is *coherent* with local charts $P_X \to \mathcal{O}_X$ with P *integral*.
- We say that a logarithmic structure is *fine and saturated* (or fs) if it is coherent with local charts $P_X \to \mathcal{O}_X$ with P *integral and saturated*.

Definition (The characteristic sheaf)

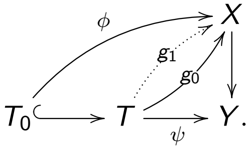
Given a logarithmic structure $X = (\underline{X}, M)$, the quotient sheaf $\overline{M} := M/\mathcal{O}_{\underline{X}}^{\times}$ is called the *characteristic sheaf* of X.

The characteristic sheaf records the combinatorics of a logarithmic structure, especially for fs logarithmic structures.

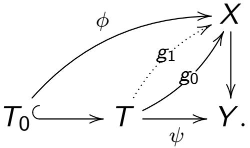
Differentials

Say $T_0 = \operatorname{Spec} k$ and $T = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$, and consider a morphism $X \to Y$.

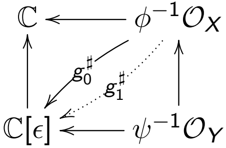
We contemplate the following diagram:

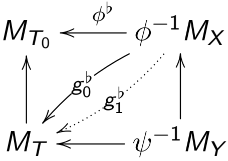


Differentials (continued)

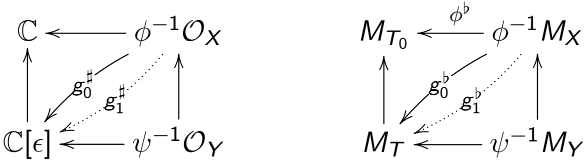


This translates to a diagram of groups and a diagram of monoids





Differentials (continued)



The difference $g_1^\sharp - g_0^\sharp$ is a derivation $\phi^{-1}\mathcal{O}_X \xrightarrow{d} \epsilon\mathbb{C} \simeq \mathbb{C}$
It comes from the sequence

$$0 \to J \to \mathcal{O}_{\underline{T}} \to \mathcal{O}_{\underline{T}_0} \to 0.$$

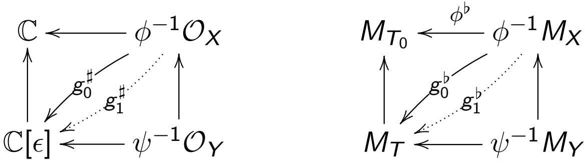
The multiplicative analogue

$$1 \to (1+J) \to \mathcal{O}_{\underline{T}}^\times \to \mathcal{O}_{\underline{T}_0}^\times \to 1$$

means, if all the logarithmic structures are integral,

$$1 \to (1+J) \to M_T \to M_{T_0} \to 1.$$

Differentials (continued)



$$1 \to (1+J) \to M_T \to M_{T_0} \to 1$$

means that we can take the "difference"

$$g_1^\flat(m) = (1 + D(m)) + g_0^\flat(m).$$

Namely $D(m) =$ "$g_1^\flat(m) - g_0^\flat(m)$" $\in J$.

Key properties:

- $D(m_1 + m_2) = D(m_1) + D(m_2)$
- $D|_{\psi^{-1}M_Y} = 0$
- $\alpha(m) \cdot D(m) = d(\alpha(m))$,

in other words,

$$D(m) \quad = \quad d \log\, (\alpha(m)),$$

which justifies the name of the theory.

Definition

A logarithmic derivation:

$$\begin{aligned} d : \mathcal{O} \quad &\to \quad J; \\ D : M \quad &\to \quad J \end{aligned}$$

satisfying the above.

Logarithmic derivations

Definition

A logarithmic derivation:

$$\begin{aligned} d : \mathcal{O} &\to J; \\ D : M &\to J \end{aligned}$$

satisfying the above.

The universal derivation:

$$d : \mathcal{O} \to \Omega^1_{\underline{X}/\underline{Y}} \quad = \quad \mathcal{O} \otimes_{\mathbb{Z}} \mathcal{O}/\text{relations}$$

The universal logarithmic derivation takes values in

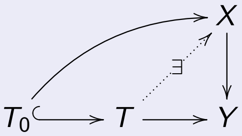
$$\Omega^1_{X/Y} = \left(\Omega^1_{\underline{X}/\underline{Y}} \oplus (\mathcal{O} \otimes_{\mathbb{Z}} M^{\mathrm{gp}})\right)/\text{relations}$$

Smoothness

Definition

We define a morphism $X \to Y$ of *fine logarithmic schemes* to be *logarithmically smooth* if

1. $\underline{X} \to \underline{Y}$ is locally of finite presentation, and
2. For T_0 fine and affine and $T_0 \subset T$ strict square-0 embedding, given



   there exists a lifting as indicated.

The morphism is *logarithmically étale* if the lifting in (2) is unique.

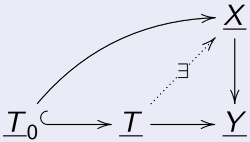
Strict smooth morphisms

Lemma

If $X \to Y$ is strict and $\underline{X} \to \underline{Y}$ smooth then $X \to Y$ is logarithmically smooth.

Proof.

There is a lifting



since $\underline{X} \to \underline{Y}$ smooth, and the lifting of morphism of monoids comes by the universal property of pullback. ♠

Combinatirially smooth morphisms

Proposition

Say P, Q are finitely generated integral *monoids, R a ring, $Q \to P$ a monoid homomorphism.*
Write $X = \mathrm{Spec}(P \to R[P])$ and $Y = \mathrm{Spec}(Q \to R[Q])$.
Assume

- $\mathrm{Ker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ *is finite and with order invertible in R,*
- $\mathrm{TorCoker}(Q^{\mathrm{gp}} \to P^{\mathrm{gp}})$ *has order invertible in R.*

Then $X \to Y$ is logarithmically smooth.
If also the cokernel is finite then $X \to Y$ is logarithmically étale.

(proof on board!)

Key examples

- Dominant toric morphisms
- Nodal curves.
- Marked nodal curves.
- $\operatorname{Spec}\mathbb{C}[t] \to \operatorname{Spec}\mathbb{C}[s]$ given by $s = t^2$
- $\operatorname{Spec}\mathbb{C}[x, y] \to \operatorname{Spec}\mathbb{C}[t]$ given by $t = x^m y^n$
- $\operatorname{Spec}\mathbb{C}[x, y] \to \operatorname{Spec}\mathbb{C}[x, z]$ given by $z = xy$.
- $\operatorname{Spec}(\mathbb{N} \to \mathbb{C}[\mathbb{N}]) \to \operatorname{Spec}((\mathbb{N}\smallsetminus 1) \to \mathbb{C}[(\mathbb{N}\smallsetminus 1)])$.

Integral morphisms

Disturbing feature: the last two examples are not flat. Which ones are flat?
We define a monoid homomorphism $Q \to P$ to be *integral* if

$$\mathbb{Z}[Q] \to \mathbb{Z}[P]$$

is flat.
A morphism $f : X \to Y$ of logarithmic schemes is *integral* if for every geometric point x of X the homomorphism

$$(f^{-1}\overline{M}_Y)_x \to (\overline{M}_X)_x$$

of characteristic sheaves is integral.

Characterization of logarithmic smoothness

Theorem (K. Kato)

X, Y fine, $Q_Y \to M_Y$ a chart.
Then $X \to Y$ is logarithmically smooth iff there are extensions to local charts

$$(P_X \to M_X, Q_Y \to M_Y, Q \to P)$$

for $X \to Y$ such that

- *$Q \to P$ combinatorially smooth, and*
- *$\underline{X} \to \underline{Y} \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P]$ is smooth*

One direction:

$$\begin{array}{ccccc} \underline{X} & \longrightarrow & \underline{Y} \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P] & \longrightarrow & \operatorname{Spec} \mathbb{Z}[P] \\ & & \downarrow & & \downarrow \\ & & \underline{Y} & \longrightarrow & \operatorname{Spec} \mathbb{Z}[Q] \end{array}$$

Deformations

Proposition (K. Kato)

If $X_0 \to Y_0$ is logarithmically smooth, $Y_0 \subset Y$ a strict square-0 extension, then locally X_0 can be lifted to a smooth $X \to Y$.

Sketch of proof: locally $X_0 \to X_0' \to Y_0$, where

$$X_0' = Y_0 \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P].$$

So $X_0' \to Y_0$ is combinatorially smooth, and automatically provided a deformation to

$$Y \times_{\operatorname{Spec} \mathbb{Z}[Q]} \operatorname{Spec} \mathbb{Z}[P],$$

and $X_0 \to X_0'$ is strict and smooth so deforms by the classical result.

Kodaira-Spencer theory

Theorem (K. Kato)

Let Y_0 be artinian, $Y_0 \subset Y$ a strict square-0 extension with ideal J, and $f_0 : X_0 \to Y_0$ logarithmically smooth. Then

- *There is a canonical element $\omega \in H^2(X_0, T_{X_0/Y_0} \otimes f_0^* J)$ such that a logarithmically smooth deformation $X \to Y$ exists if and only if $\omega = 0$.*
- *If $\omega = 0$, then isomorphism classes of such $X \to Y$ correspond to elements of a torsor under $H^1(X_0, T_{X_0/Y_0} \otimes f_0^* J)$.*
- *Given such deformation $X \to Y$, its automorphism group is $H^0(X_0, T_{X_0/Y_0} \otimes f_0^* J)$.*

Corollary

Logarithmically smooth curves are unobstructed.

Saturated morphisms

Recall that the monoid homomorphism $\mathbb{N} \xrightarrow{\cdot 2} \mathbb{N}$ gives an integral logarithmically étale map with non-reduced fibers.

Definition

- An integral $Q \to P$ of saturated monoids is said to be *saturated* if $\mathrm{Spec}(P \to \mathbb{Z}[P]) \to \mathrm{Spec}(Q \to \mathbb{Z}[Q])$ has reduced fibers.
- An integral morphism $X \to Y$ of fs logarithmic schemes is *saturated* if it has a saturated chart.

This guarantees that if $X \to Y$ is logarithmically smooth, then the fibers are reduced.

Heros:

- Olsson; Chen, Gillam, Huang, Satriano, Sun; Gross - Siebert
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- K. Kato
- F. Kato

Log curves

Definition

A *log curve* is a morphism $f : X \to S$ of fs logarithmic schemes satisfying:

- f is logarithmically smooth,
- f is integral, i.e. flat,
- f is saturated, i.e. has reduced fibers, and
- the fibers are curves i.e. pure dimension 1 schemes.

Theorem (F. Kato)

Assume $\pi : X \to S$ is a log curve. Then

- *Fibers have at most nodes as singularities*
- *étale locally on S we can choose disjoint sections $s_i : \underline{S} \to \underline{X}$ in the nonsingular locus $\underline{X}_0$ of $\underline{X}/\underline{S}$ such that*
    - *Away from s_i we have that $X^0 = \underline{X}_0 \times_{\underline{S}} S$, so π is strict away from s_i*
    - *Near each s_i we have a strict étale*

      $$X^0 \to S \times \mathbb{A}^1$$

      with the standard divisorial logarithmic structure on $\mathbb{A}^1$.
    - *étale locally at a node $xy = f$ the log curve X is the pullback of*

      $$\mathrm{Spec}(\mathbb{N}^2 \to \mathbb{Z}[\mathbb{N}^2]) \to Spec(\mathbb{N} \to \mathbb{Z}[\mathbb{N}])$$

      where $\mathbb{N} \to \mathbb{N}^2$ is the diagonal. Here the image of $1 \in \mathbb{N}$ in $\mathcal{O}_S$ is f and the generators of $\mathbb{N}^2$ map to x and y.

Stable log curves

Definition

A *stable log curve* $X \to S$ is:

- a log curve $X \to S$,
- sections $s_i : \underline{S} \to \underline{X}$ for $i = 1, \ldots, n$,

such that

- $(\underline{X} \to \underline{S}, s_i)$ is stable,
- the log structure is strict away from sections and singularities of fibers, and "divisorial along the sections".

Moduli of stable log curves

We define a category $\overline{\mathcal{M}}_{g,n}^{\log}$ of stable log curves: objects are log (g,n)-curves $X \to S$ and arrows are fiber diagrams compatible with sections

$$\begin{array}{ccc} X_1 & \longrightarrow & X_2 \\ \downarrow & & \downarrow \\ S_2 & \longrightarrow & S_2 \end{array}.$$

There is a forgetful functor

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}^{\log} & \longrightarrow & \mathfrak{LogSch}^{\mathrm{fs}} \\ (X \to S) & \mapsto & S. \end{array}$$

So $\overline{\mathcal{M}}_{g,n}^{\log}$ is a category fibered in groupoids over $\mathfrak{LogSch}^{\mathrm{fs}}$.

Moduli of stable log curves (continued)

We also have a forgetful functor

$$\begin{array}{ccc} \overline{\mathcal{M}}_{g,n}^{\log} & \longrightarrow & \overline{\mathcal{M}}_{g,n} \\ (X \to S) & \mapsto & (\underline{X} \to \underline{S}) \end{array}$$

Note that the Deligne–Knudsen–Mumford moduli stack $\overline{\mathcal{M}}_{g,n}$ has a natural logarithmic smooth structure $M_{\Delta_{g,n}}$ given by the boundary divisor. As such it represents a category fibered in groupoids $(\overline{\mathcal{M}}_{g,n}, M_{\Delta_{g,n}})$ over $\mathfrak{LogSch}^{\mathsf{fs}}$.

Theorem (F. Kato)

$$\overline{\mathcal{M}}_{g,n}^{\log} \simeq (\overline{\mathcal{M}}_{g,n}, M_{\Delta_{g,n}}).$$

(Proof sketch on board)

Minimality

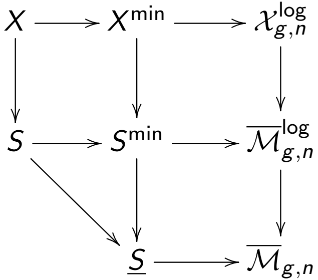
Given a stable curve $\underline{X} \to \underline{S}$ we define

$$X^{\min} = \underline{X} \times_{\overline{\mathcal{M}}_{g,n+1}} \overline{\mathcal{M}}_{g,n+1}^{\log} \qquad \text{and} \qquad S^{\min} = \underline{S} \times_{\overline{\mathcal{M}}_{g,n}} \overline{\mathcal{M}}_{g,n}^{\log}.$$

The logarithmic structures $X^{\min} \to S^{\min}$ are called the *minimal* or *basic* logarithmic structures on a log curve.
We write

$$S^{\min} = (\underline{S}, M_{X/S}^{S}) \qquad \text{and} \qquad X^{\min} = (\underline{X}, M_{X/S}^{X}).$$

Fundamental diagram



- $\overline{\mathcal{M}}_{g,n}^{\log}$ parametrizes stable log curves over $\mathfrak{LogSch}^{\text{fs}}$
- $\overline{\mathcal{M}}_{g,n}$ parametrizes minimal stable log curves over $\mathfrak{Sch}$.

Stable logarithmic maps

Definition

A **stable logarithmic map** is a diagram

$$\begin{array}{ccc} C & \xrightarrow{f} & X \\ {\scriptstyle s_i}\uparrow\downarrow{\scriptstyle \pi} & & \\ S & & \end{array}$$

where

- $(C/S, s_i)$ is a prestable log curve, and
- in fibers $\mathrm{Aut}(\underline{C}_s \to \underline{X}, s_i)$ is finite.

contact orders

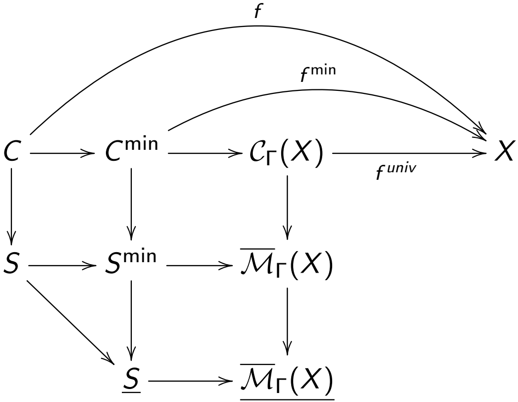
Apart from the underlying discrete data $\underline{\Gamma} = (g, \beta, n)$, a stable logarithmic map has *contact orders* c_i at the marked points.
At each such point the logarithmic structure at C has a factor $\mathbb{N}$, and the contact order is the homomorphism $f^*M_X \stackrel{c_i}{\to} \mathbb{N}$ at that marked point.
We collect the numerical data under the umbrella $\Gamma = (g, \beta, c_i)$.

Stable logarithmic maps (continued)

Theorem (Gross-Siebert, Chen, $\aleph$-Chen-Marcus-Wise)

Let X be projective logarithmically scheme. Stable logarithmic maps to X form a logarithmic Deligne–Mumford stack $\overline{\mathcal{M}}_\Gamma(X)$. It is finite and representable over $\overline{\mathcal{M}}_{\underline{\Gamma}}(\underline{X})$.

Fundamental diagram



We are in search of a moduli stack $\underline{\overline{\mathcal{M}}_\Gamma(X)}$ parametrizing *minimal* stable logarithmic maps over $\mathfrak{Sch}$.
As such it comes with a logarithmic structure $\overline{\mathcal{M}}_\Gamma(X)$ which parametrizes *all* stable logarithmic maps over $\mathfrak{LogSch}^{\mathrm{fs}}$.

Stable logarithmic maps (continued)

This requires two steps:

- first find a morphism from $(C \to S, f : C \to X)$ to a *minimal* object $(C^{\min} \to S^{\min}, f^{\min} : C^{\min} \to X)$.
- then show that the object $(C^{\min} \to S^{\min}, f^{\min} : C^{\min} \to X)$ has a versal deformation space, whose fibers are also minimal.

Minimal stable logarithmic maps

We consider X toric and a stable logarithmic map $(C/S, f : C \to X)$ over a P-logarithmic point S.
We wish to find a minimal Q-logarithmic point and a logarithmic map over it through which our object factors.
We might as well first pull back and replace S by a standard, $P = \mathbb{N}$-logarithmic point!
The curve C has components C_i with generic points η_i corresponding to vertices in the dual graph, and nodes q_j with local equations $xy = g_j$ corresponding to edges in the dual graph.

At the generic points

The map f sends η_i to some stratum X_i of X with cone σ_i having lattice $N_i \subset \sigma_i$.
Departing from toric conventions we denote $M_i = N_i^\vee = \underline{\mathrm{Hom}}(N_i, \mathbb{N})$.
Since the logarithmic structure of C at η_i is the pullback of the structure on S, we have a map $f_i^\flat : M_i \to P$.
It can dually be viewed as a map $P^\vee \to N_i$, or an element $v_i \in N_i$.
If that were all we had, our final object would be $Q^\vee = \prod N_i$, and dually the initial monoid $Q = \oplus M_i$.
But the nodes impose crucial conditions.

At the nodes

At a node q with branches η_q^1, η_q^2 we similarly have a map

$$f_q^\flat : M_q \to P \oplus^{\mathbb{N}} \mathbb{N}^2.$$

Unfortunately it is unnatural to consider maps into a coproduct, and we give an alterante description of

$$P \oplus^{\mathbb{N}} \mathbb{N}^2 = P\langle \log x, \log y \rangle / (\log x + \log y = \rho_q)$$

where $\rho_q = \log g_q \in P$.
Recall that the stalk of a sheaf at a point q maps, via a "generization map", to the stalk at any point specializing to q, such as η_q^1, η_q^2.

At the nodes (continued)

$$P \oplus^{\mathbb{N}} \mathbb{N}^2 = P\langle \log x, \log y\rangle/(\log x + \log y = \rho_q)$$

The map to the stalk at η_q^1 where $x = 0$ sends $\log y \mapsto 0$, and so $\log x \mapsto \rho_q$.
The map to the stalk at η_q^2 where $y = 0$ sends $\log x \mapsto 0$, and so $\log y \mapsto \rho_q$.
This means that we have a monoid homomorphism, which is clearly injective,

$$P \oplus^{\mathbb{N}} \mathbb{N}^2 \to P \times P.$$

Its image is precisely the set of pairs

$$\{(p_1, p_2) | p_2 - p_1 \in \mathbb{Z}\rho_q\}$$

At the nodes (continued)

$$P \oplus^{\mathbb{N}} \mathbb{N}^2 = \{(p_1, p_2) | p_2 - p_1 \in \mathbb{Z}\rho_q\}$$

- $f_q^\flat : M_q \to P \times P$,
- $(p_2 - p_1) \circ f_q^\flat : M_q \to \mathbb{Z}\rho_q \subset P^{\mathrm{gp}}$.

Or better: we have $u_q : M_q \to \mathbb{Z}$ such that

$$(p_2 - p_1) \circ f_q^\flat(m) = u_q(m) \cdot \rho_q. \qquad (0.0.1)$$

Putting nodes and generic points together

The maps $p_1 \circ f_q^\flat : M_q \to P$ and $p_2 \circ f_q^\flat : M_q \to P$, since they come from maps of sheaves, are compatible with generization maps.

$p_1 \circ f_q^\flat : M_q \to P$ is the composition $M_q \to M_{\eta_q^1} \to P$

$p_2 \circ f_q^\flat : M_q \to P$ is the composition of $M_q \to M_{\eta_q^2} \to P$

the data of $p_1 \circ f_q^\flat$ and $p_2 \circ f_q^\flat$ is already determined by the data at the generic points η_i of the curve.

Putting nodes and generic points together (continued)

The *only* data the node provides is the element $\rho_q \in P$ and homomorphism $u_q : M_q \to \mathbb{Z}$, in such a way that equation

$$(p_2 - p_1) \circ f_q^\flat(m) = u_q(m) \cdot \rho_q. \qquad (0.0.2)$$

holds.

$$Q_f = \left(\left(\prod_\eta M_{\sigma_\eta} \times \prod_q \mathbb{N} \right) \Big/ R \right)^{sat}$$

where R is generated by all the relations implied by equation (**??**)

Putting nodes and generic points together (continued)

$$Q_f = \left(\left(\prod_\eta M_{\sigma_\eta} \times \prod_q \mathbb{N}\right) \Big/ R \right)^{sat}$$

It is quite a bit more natural to describe the dual lattice

$$Q_f^\vee = \left\{ ((v_\eta),(e_\eta)) \in \prod_\eta N_{\sigma_\eta} \times \prod_q \mathbb{N} \;\middle|\; \begin{array}{c} \forall\ \eta_q^1 \xrightarrow{\quad q \quad} \eta_q^2 \\ v_q^1 - v_q^2 = e_j u_q \end{array} \right\}.$$

Tropical interpretation

Given a map f over an $\mathbb{N}$-point we have a graph in $\Sigma(X)$ with

- vertices $v_i \in N_{\sigma_\eta} \subset \sigma_\eta$
- edges proportional to $u_q \in N_q^{\mathrm{gp}}$ such that $v_q^1 - v_q^2 = e_j u_q$

this means

- The equations $v_q^1 - v_q^2 = e_j u_q$ define the cone of all such graphs
- $Q_f^\vee$ is the integer lattice in that cone.

Theorem (Gross-Siebert)

The minimal object exists, with characteristic sheaf Q_f, dual to the lattice in the corresponding space of tropical curves.