Logarithmic Geometry and Moduli
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Dan Abramovich
Brown University

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Moduli of curves

$\mathcal{M}_g$ - a quasiprojective variety.

*Working with a non-complete moduli space is like keeping change in a pocket with holes*

Angelo Vistoli
Deligne–Mumford

- $\mathcal{M}_g \subset \overline{\mathcal{M}}_g$ - moduli of stable curves, a modular compactification.
- allow only nodes as singularities
- What’s so great about nodes?
- One answer: from the point of view of logarithmic geometry, these are the logarithmically smooth curves.
Logarithmic structures

Definition

A pre logarithmic structure is

\[ X = (X, M \xrightarrow{\alpha} \mathcal{O}_X) \quad \text{or just} \quad (X, M) \]

such that

- \( X \) is a scheme - the underlying scheme
- \( M \) is a sheaf of monoids on \( X \), and
- \( \alpha \) is a monoid homomorphism, where the monoid structure on \( \mathcal{O}_X \) is the multiplicative structure.

Definition

It is a logarithmic structure if \( \alpha : \alpha^{-1}\mathcal{O}_X^* \to \mathcal{O}_X^* \) is an isomorphism.
“Trivial” examples

Examples

- \((X, \mathcal{O}_X^* \hookrightarrow \mathcal{O}_X)\), the trivial logarithmic structure. We sometimes write just \(X\) for this structure.
- \((X, \mathcal{O}_X \xrightarrow{\sim} \mathcal{O}_X)\), looks as easy but surprisingly not interesting, and
- \((X, \mathbb{N} \xrightarrow{\alpha} \mathcal{O}_X)\), where \(\alpha\) is determined by an arbitrary choice of \(\alpha(1)\). This one is important but only pre-logarithmic.

\(^{a}\text{not according to Dhruv!}\)
The associated logarithmic structure

You can always fix a pre-logarithmic structure:

\[
\begin{align*}
\alpha^{-1}O^* & \rightarrow M \\
O^* & \rightarrow M^a \\
O & \rightarrow O
\end{align*}
\]
Key examples

Example (Divisorial logarithmic structure)
Let $X, D \subset X$ be a variety with a divisor. We define $M_D \hookrightarrow \mathcal{O}_X$:

$$M_D(U) = \left\{ f \in \mathcal{O}_X(U) \mid f_{U \setminus D} \in \mathcal{O}_X^\times(U \setminus D) \right\}.$$

This is particularly important for normal crossings divisors and toric divisors - these will be logarithmically smooth structures.
Example (Standard logarithmic point)

Let $k$ be a field,

$$\mathbb{N} \oplus k^\times \to k$$

$$(n, z) \mapsto z \cdot 0^n$$

defined by sending $0 \mapsto 1$ and $n \mapsto 0$ otherwise.

Works with $P$ a monoid with $P^\times = 0$, giving the $P$-logarithmic point. This is what you get when you restrict the structure on an affine toric variety associated to $P$ to the maximal ideal generated by $\{p \neq 0\}$. 
Morphisms

A morphism of (pre)-logarithmic schemes $f : X \to Y$ consists of

- $f : X \to Y$
- A homomorphism $f^\flat$ making the following diagram commutative:

$$
\begin{array}{ccc}
M_X & \xleftarrow{f^\flat} & f^{-1}M_Y \\
\downarrow{\alpha_X} & & \downarrow{\alpha_Y} \\
O_X & \xleftarrow{f^\#} & f^{-1}O_Y \\
\end{array}
$$
Definition (Inverse image)

Given \( f : X \to Y \) and \( Y = (Y, M_Y) \) define the \textit{pre-logarithmic inverse image} by composing

\[
f^{-1}M_Y \to f^{-1}O_Y \xrightarrow{f^\#} O_X
\]

and then the \textit{logarithmic inverse image} is defined as

\[
f^* M_Y = (f^{-1} M_Y)^a.
\]

This is the universal logarithmic structure on \( X \) with commutative

\[
(X, f^* M_Y) \to Y
\]

\[
\downarrow \quad \downarrow
\]

\[
X \to Y
\]

\( X \to Y \) is \textbf{strict} if \( M_X = f^* M_Y \).
Definition (Fibered products)

The fibered product $X \times_Z Y$ is defined as follows:

- $X \times_Z Y = X \times_Z Y$
- If $N$ is the pushout of

\[ \begin{array}{c}
\pi_{X}^{-1} M_{X} \\
\downarrow \\
\pi_{Z}^{-1} M_{Z} \\
\downarrow \\
\pi_{Y}^{-1} M_{Y}
\end{array} \]

then the log structure on $X \times_Z Y$ is defined by $N^a$. 

Definition (The spectrum of a Monoid algebras)

Let $P$ be a monoid, $R$ a ring. We obtain a monoid algebra $R[P]$ and a scheme $\underline{X} = \text{Spec } R[P]$. There is an evident monoid homomorphism $P \to R[P]$ inducing sheaf homomorphism $P_\underline{X} \to \mathcal{O}_\underline{X}$, a pre-logarithmic structure, giving rise to a logarithmic structure

$$(P_\underline{X})^a \to \mathcal{O}_\underline{X}.$$ 

This is a basic example. It deserves a notation:

$$\underline{X} = \text{Spec}(P \to R[P]).$$

The most basic example is $X_0 = \text{Spec}(P \to \mathbb{Z}[P]).$

The morphism $\underline{f} : \text{Spec}(R[P]) \to \text{Spec}(\mathbb{Z}[P])$ gives

$$\underline{X} = \underline{X} \times_{X_0} X_0.$$
Charts

A chart for $X$ is given by a monoid $P$ and a sheaf homomorphism $P_X \to O_X$ to which $X$ is associated.
This is the same as a strict morphism $X \to \text{Spec}(P \to \mathbb{Z}[P])$.

Given a morphism of logarithmic schemes $f : X \to Y$, a chart for $f$ is a triple

$$(P_X \to M_X, Q_Y \to M_Y, Q \to P)$$

such that

- $P_X \to M_X$ and $Q_Y \to M_Y$ are charts for $M_X$ and $M_Y$, and
- the diagram

$$
\begin{array}{ccc}
Q_X & \longrightarrow & f^{-1}M_Y \\
\downarrow & & \downarrow \\
P_X & \longrightarrow & M_X
\end{array}
$$

is commutative.
Types of logarithmic structures

- We say that \((X, M_X)\) is **coherent** if étale locally at every point there is a **finitely generated** monoid \(P\) and a local chart \(P_X \to \mathcal{O}_X\) for \(X\).

- A monoid \(P\) is **integral** if \(P \to \mathcal{P}_{\text{gp}}\) is injective.

- It is **saturated** if integral and whenever \(p \in \mathcal{P}_{\text{gp}}\) and \(m \cdot p \in P\) for some integer \(m > 0\) then \(p \in P\). i.e., not like \(\{0, 2, 3, \ldots\}\).

- We say that a logarithmic structure is **fine** if it is **coherent** with local charts \(P_X \to \mathcal{O}_X\) with \(P\) **integral**.

- We say that a logarithmic structure is **fine and saturated** (or **fs**) if it is coherent with local charts \(P_X \to \mathcal{O}_X\) with \(P\) **integral and saturated**.
Definition (The characteristic sheaf)

Given a logarithmic structure $X = (\mathcal{X}, M)$, the quotient sheaf $\overline{M} := M / \mathcal{O}_X^\times$ is called the characteristic sheaf of $X$.

The characteristic sheaf records the combinatorics of a logarithmic structure, especially for fs logarithmic structures.
Smoothness

**Definition**

We define a morphism \( X \to Y \) of *fine logarithmic schemes* to be *logarithmically smooth* if

1. \( X \to Y \) is locally of finite presentation, and
2. For \( T_0 \) fine and affine and \( T_0 \subset T \) strict square-0 embedding, given

\[
\begin{array}{ccc}
X & \to & Y \\
\uparrow & & \downarrow \\
T_0 & \to & T & \to & Y
\end{array}
\]

there exists a lifting as indicated.

The morphism is *logarithmically étale* if the lifting in (2) is unique.
**Lemma**

If $X \to Y$ is strict and $X \to Y$ smooth then $X \to Y$ is logarithmically smooth.

**Proof.**

There is a lifting

Since $X \to Y$ smooth, and the lifting of morphism of monoids comes by the universal property of pullback.
Combinatorially smooth morphisms

Proposition

Say $P, Q$ are finitely generated integral monoids, $R$ a ring, $Q \to P$ a monoid homomorphism.
Write $X = \text{Spec}(P \to R[P])$ and $Y = \text{Spec}(Q \to R[Q])$.
Assume

- $\text{Ker}(Q^{\text{gp}} \to P^{\text{gp}})$ is finite and with order invertible in $R$,
- $\text{TorCoker}(Q^{\text{gp}} \to P^{\text{gp}})$ has order invertible in $R$.

Then $X \to Y$ is logarithmically smooth.
If also the cokernel is finite then $X \to Y$ is logarithmically étale.
Key examples

- Dominant toric morphisms
- Nodal curves.
- Marked nodal curves.
- Spec $\mathbb{C}[t] \to \text{Spec } \mathbb{C}[s]$ given by $s = t^2$
- Spec $\mathbb{C}[x, y] \to \text{Spec } \mathbb{C}[t]$ given by $t = x^m y^n$
- Spec $\mathbb{C}[x, y] \to \text{Spec } \mathbb{C}[x, z]$ given by $z = xy$.
- Spec$(\mathbb{N} \to \mathbb{C}[\mathbb{N}]) \to \text{Spec}((\mathbb{N}\backslash 1) \to \mathbb{C}[(\mathbb{N}\backslash 1)])$. 
Integral morphisms

Disturbing feature: the last two examples are not flat. Which ones are flat? We define a monoid homomorphism $Q \to P$ to be *integral* if

$$\mathbb{Z}[Q] \to \mathbb{Z}[P]$$

is flat.\(^1\)

A morphism $f : X \to Y$ of logarithmic schemes is *integral* if for every geometric point $x$ of $X$ the homomorphism

$$(f^{-1}M_Y)_x \to (M_X)_x$$

of characteristic sheaves is integral.

\(^1\)Has natural universal property
Theorem (K. Kato)

Let $X, Y$ be fine, and $Q_Y \to M_Y$ a chart. Then $X \to Y$ is logarithmically smooth iff there are extensions to local charts

$$(P_X \to M_X, Q_Y \to M_Y, Q \to P)$$

for $X \to Y$ such that

- $Q \to P$ combinatorially smooth, and
- $X \to Y \times \text{Spec} \mathbb{Z}[Q] \text{ Spec} \mathbb{Z}[P]$ is smooth

One direction:
**Proposition (K. Kato)**

> If $X_0 \to Y_0$ is logarithmically smooth, $Y_0 \subset Y$ a strict square-0 extension, then locally $X_0$ can be lifted to a smooth $X \to Y$.

**Sketch of proof:** locally $X_0 \to X'_0 \to Y_0$, where

$$X'_0 = Y_0 \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P].$$

So $X'_0 \to Y_0$ is combinatorially smooth, and automatically provided a deformation to

$$Y \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P],$$

and $X_0 \to X'_0$ is strict and smooth so deforms by the classical result.
Kodaira-Spencer theory

**Theorem (K. Kato)**

Let $Y_0$ be artinian, $Y_0 \subset Y$ a strict square-0 extension with ideal $J$, and $f_0 : X_0 \to Y_0$ logarithmically smooth. Then

- There is a canonical element $\omega \in H^2(X_0, T_{X_0/Y_0} \otimes f_0^* J)$ such that a logarithmically smooth deformation $X \to Y$ exists if and only if $\omega = 0$.
- If $\omega = 0$, then isomorphism classes of such $X \to Y$ correspond to elements of a torsor under $H^1(X_0, T_{X_0/Y_0} \otimes f_0^* J)$.
- Given such deformation $X \to Y$, its automorphism group is $H^0(X_0, T_{X_0/Y_0} \otimes f_0^* J)$.

**Corollary**

Logarithmically smooth curves are unobstructed.
Saturated morphisms

Recall that the monoid homomorphism \( \mathbb{N} \rightarrow \mathbb{N} \) gives an integral logarithmically étale map with non-reduced fibers.

**Definition**

- An integral \( Q \rightarrow P \) of saturated monoids is said to be **saturated** if \( \text{Spec}(P \rightarrow \mathbb{Z}[P]) \rightarrow \text{Spec}(Q \rightarrow \mathbb{Z}[Q]) \) has reduced fibers.\(^a\)
- An integral morphism \( X \rightarrow Y \) of fs logarithmic schemes is **saturated** if it has a saturated chart.

\(^a\)Has natural universal property

This guarantees that if \( X \rightarrow Y \) is logarithmically smooth, then the fibers are reduced.
Log curves

Definition

A \textit{log curve} is a morphism $f : X \to S$ of fs logarithmic schemes satisfying:

- $f$ is logarithmically smooth,
- $f$ is integral, i.e. flat,
- $f$ is saturated, i.e. has reduced fibers, and
- the fibers are curves i.e. pure dimension 1 schemes.
Theorem (F. Kato)

Assume \( \pi : X \to S \) is a log curve. Then

- \textbf{fibers have at most nodes as singularities}
- \textbf{étale locally on } \( S \) \textbf{we can choose disjoint sections } \( s_i : S \to X \) \textbf{in the nonsingular locus } \( X_0 \) \textbf{of } \( X/S \) \textbf{such that}
  - Away from \( s_i \) we have that \( X^0 = X_0 \times_S S \), so \( \pi \) is strict away from \( s_i \)
  - Near each \( s_i \) we have a strict \textbf{étale}

\[
X^0 \to S \times \mathbb{A}^1
\]

with the standard divisorial logarithmic structure on \( \mathbb{A}^1 \).
- \textbf{étale locally at a node } \( xy = f \) \textbf{the log curve } \( X \) \textbf{is the pullback of}

\[
\text{Spec}(\mathbb{N}^2 \to \mathbb{Z}[\mathbb{N}^2]) \to \text{Spec}(\mathbb{N} \to \mathbb{Z}[\mathbb{N}])
\]

where \( \mathbb{N} \to \mathbb{N}^2 \) \textbf{is the diagonal}. Here the image of } \( 1 \in \mathbb{N} \) \textbf{in } \( \mathcal{O}_S \) \textbf{is } \( f \) \textbf{and the generators of } \( \mathbb{N}^2 \) \textbf{map to } \( x \) \textbf{and } \( y \).
A stable log curve $X \to S$ is:

- a log curve $X \to S$,
- sections $s_i : S \to X$ for $i = 1, \ldots, n$,

such that

- $(X \to S, s_i)$ is stable,
- the log structure is strict away from sections and singularities of fibers, and “divisorial along the sections”.

**Definition**

**Stable log curves**
Moduli of stable log curves

We define a category $\mathcal{M}_{g,n}^{\text{log}}$ of stable log curves: objects are log $(g, n)$-curves $X \to S$ and arrows are fiber diagrams compatible with sections

$$
\begin{array}{ccc}
X_1 & \longrightarrow & X_2 \\
\downarrow & & \downarrow \\
S_2 & \longrightarrow & S_2
\end{array}
$$

There is a forgetful functor

$$
\mathcal{M}_{g,n}^{\text{log}} \longrightarrow \text{LogSch}^{\text{fs}}
$$

$$(X \to S) \mapsto S.$$

So $\mathcal{M}_{g,n}^{\text{log}}$ is a category fibered in groupoids over $\text{LogSch}^{\text{fs}}$. 
Moduli of stable log curves (continued)

We also have a forgetful functor

\[ \mathcal{M}^\text{log}_{g,n} \longrightarrow \mathcal{M}_{g,n} \]

\[(X \to S) \mapsto (X \to S)\]

Note that the Deligne–Knudsen–Mumford moduli stack \( \mathcal{M}_{g,n} \) has a natural logarithmic smooth structure \( \mathcal{M}^{\Delta}_{g,n} \) given by the boundary divisor. As such it represents a category fibered in groupoids \((\mathcal{M}_{g,n}, \mathcal{M}^{\Delta}_{g,n})\) over \(\text{LogSch}^{\text{fs}}\).

**Theorem (F. Kato)**

\[ \mathcal{M}^\text{log}_{g,n} \simeq (\mathcal{M}_{g,n}, \mathcal{M}^{\Delta}_{g,n}). \]
The stack $\overline{M}_{g,n}^{\log}$ is huge!

- Given a stable log curve $X \to S$ and an arbitrary log point $S'$ then $X \times S' \to S \times S'$ is a stable log curve on the same underlying scheme.
- Given $S = \text{Spec}(\mathbb{N}^2 \to \mathbb{C})$, a stable log curve $X \to S$ and a quotient $\mathbb{N}^2 \to \mathbb{N}$ we get a log point $S' = \text{Spec}(\mathbb{N} \to \mathbb{C}) \to S$ and a new stable log curve $X' \to S'$ on the same underlying scheme.
- We claim that some stable log curves $X \to S$ are more fundamental than others.
- Not all logarithmic moduli problems are so lucky! There are issues with logarithmic $\mathbb{G}_m$ and logarithmic Picard (Molcho-Wise).
Given a stable curve $X \to S$ we define

$$X^{\min} = X \times \overline{M}_{g,n+1}^{\log} \quad \text{and} \quad S^{\min} = S \times \overline{M}_{g,n}^{\log}.$$

The logarithmic structures $X^{\min} \to S^{\min}$ are called the *minimal* or *basic* logarithmic structures on a log curve.

We write

$$S^{\min} = (S, M^S_{X/S}) \quad \text{and} \quad X^{\min} = (X, M^X_{X/S}).$$
Fundamental diagram

\[
\begin{array}{ccc}
X & \longrightarrow & X_{\min} \\
\downarrow & & \downarrow \\
S & \longrightarrow & S_{\min} \\
\downarrow & & \downarrow \\
S & \longrightarrow & \overline{M}_{g,n}
\end{array}
\]

- $\overline{M}_{g,n}^{\log}$ parametrizes stable log curves over $\text{LogSch}^{fs}$
- $\overline{M}_{g,n}$ parametrizes minimal stable log curves over $\text{Sch}$.
Stable logarithmic maps

Definition

A **stable logarithmic map** is a diagram

\[
\begin{array}{ccc}
C & \xrightarrow{f} & X \\
\downarrow s_i & & \downarrow \pi \\
S & & 
\end{array}
\]

where

- \((C/S, s_i)\) is a prestable log curve, and
- in fibers \(\text{Aut}(C_s \to X, s_i)\) is finite.
Apart from the underlying discrete data $\Gamma = (g, \beta, n)$, a stable logarithmic map has *contact orders* $c_i$ at the marked points. At each such point the logarithmic structure at $C$ has a factor $\mathbb{N}$, and the contact order is the homomorphism $f^*M_X \xrightarrow{c_i} \mathbb{N}$ at that marked point. We collect the numerical data under the umbrella $\Gamma = (g, \beta, c_i)$. 

**contact orders**
Theorem (Gross-Siebert, Chen, \& Chen-Marcus-Wise)

Let $X$ be projective logarithmically scheme. Stable logarithmic maps to $X$ form a logarithmic Deligne–Mumford stack $\overline{M}_Γ(X)$. It is finite and representable over $\overline{M}_Γ(X)$. 
We are in search of a moduli stack $\overline{M}_\Gamma(X)$ parametrizing *minimal* stable logarithmic maps over $\mathcal{S}ch$. As such it comes with a logarithmic structure $\overline{M}_\Gamma(X)$ which parametrizes *all* stable logarithmic maps over $\log\mathcal{S}ch^{fs}$. 

![Diagram](attachment:image.png)
This requires two steps:

- first find a morphism from \((C \to S, f : C \to X)\) to a \textit{minimal} object \((C_{\text{min}} \to S_{\text{min}}, f_{\text{min}} : C_{\text{min}} \to X)\).
- then show that the object \((C_{\text{min}} \to S_{\text{min}}, f_{\text{min}} : C_{\text{min}} \to X)\) has a versal deformation space, \(S \to V\), with universal family

\[ (C_{V_{\text{min}}} \to V_{\text{min}}, f_{V_{\text{min}}} : C_{V_{\text{min}}} \to X), \]

whose fibers over any \(T \to V\) are also minimal.
Minimal stable logarithmic maps

We consider $X$ toric and a stable logarithmic map $(C/S, f : C \to X)$ over a $P$-logarithmic point $S$.
We wish to find a minimal $Q$-logarithmic point and a logarithmic map over it through which our object factors.
We might as well first pull back and replace $S$ by a standard, $P = \mathbb{N}$-logarithmic point!

Indeed if $S_P \to S$ is such a point and $S_P \to S_Q$ is its map to the minimal structure, then minimality says there is a unique $S \to S_Q$ factoring $S_P \to S_Q$.
At the generic points

The curve $C$ has components $C_i$ with generic points $\eta_i$ corresponding to vertices in the dual graph, and nodes $q_j$ with local equations $xy = g_j$ corresponding to edges in the dual graph.

- The map $f$ sends $\eta_i$ to some stratum $X_i$ of $X$ with cone $\sigma_i$ having lattice $N_i \subset \sigma_i$.
- Departing from toric conventions we denote $M_i = N_i^\vee = \text{Hom}(N_i, \mathbb{N})$.
- Since the logarithmic structure of $C$ at $\eta_i$ is the pullback of the structure on $S$, we have a map $f_i^\flat : M_i \to P$.
- It can dually be viewed as a map $P^\vee \to N_i$, or an element $v_i \in N_i$.
- If that were all we had, our final object would be $Q^\vee = \prod N_i$, and dually the initial monoid $Q = \bigoplus M_i$.

But the nodes impose crucial conditions.
At the nodes

- At a node $q$ with branches $\eta^1_q, \eta^2_q$ we similarly have a map
  \[ f^b_q : M_q \to P \oplus \mathbb{N} \mathbb{N}^2. \]

- Unfortunately it is unnatural to consider maps into a coproduct, and we give an alternative description of
  \[ P \oplus \mathbb{N} \mathbb{N}^2 = P\langle \log x, \log y \rangle / (\log x + \log y = \rho_q) \]
  where $\rho_q = \log g_q \in P$.

- Recall that the stalk of a sheaf at a point $q$ maps, via a “generalization map”, to the stalk at any point specializing to $q$, such as $\eta^1_q, \eta^2_q$.

- The stalk at either $\eta^1_q, \eta^2_q$ is $P$. 
At the nodes (continued)

\[ P \oplus \mathbb{N} \mathbb{N}^2 = P \langle \log x, \log y \rangle / (\log x + \log y = \rho_q) \]

The map to the stalk \( P \) at \( \eta_1^1 \) where \( x = 0 \) sends \( \log y \mapsto 0 \), and so \( \log x \mapsto \rho_q \).

The map to the stalk \( P \) at \( \eta_2^2 \) where \( y = 0 \) sends \( \log x \mapsto 0 \), and so \( \log y \mapsto \rho_q \).

This means that we have a monoid homomorphism, which is clearly injective,

\[ P \oplus \mathbb{N} \mathbb{N}^2 \to P \times P. \]

Its image is precisely the set of pairs

\[ \{(p_1, p_2) | p_2 - p_1 \in \mathbb{Z} \rho_q \} \]
At the nodes (continued)

\[ P \oplus^N \mathbb{N}^2 = \{(p_1, p_2) | p_2 - p_1 \in \mathbb{Z}\rho_q\} \]

- \( f^b_q : M_q \to P \times P \),
- \((p_2 - p_1) \circ f^b_q : M_q \to \mathbb{Z}\rho_q \subset P^{gp} \).

Or better: we have \( u_q : M_q \to \mathbb{Z} \) such that

\[ (p_2 - p_1) \circ f^b_q(m) = u_q(m) \cdot \rho_q. \quad (0.0.1) \]
Putting nodes and generic points together

The maps $p_1 \circ f_q^b : M_q \to P$ and $p_2 \circ f_q^b : M_q \to P$, since they come from maps of sheaves, are compatible with generization maps.

$p_1 \circ f_q^b : M_q \to P$ is the composition $M_q \to M_{\eta_1 q} \to P$

$p_2 \circ f_q^b : M_q \to P$ is the composition of $M_q \to M_{\eta_2 q} \to P$

*the data of $p_1 \circ f_q^b$ and $p_2 \circ f_q^b$ is already determined by the data at the generic points $\eta_i$ of the curve.*
Putting nodes and generic points together (continued)

The additional data the node provides is the element $\rho_q \in P$ and homomorphism $u_q : M_q \to \mathbb{Z}$, in such a way that equation

$$ (p_2 - p_1) \circ f_q^b(m) = u_q(m) \cdot \rho_q. \quad (0.0.2) $$

holds.

$$ Q_f = \left( \left( \prod_\eta M_{\sigma_\eta} \times \prod_q \mathbb{N} \right) / R \right)^{sat} $$

where the saturated submonoid $R$ is generated by all the relations implied by equation (??)
Putting nodes and generic points together (continued)

\[
Q_f = \left( \left( \prod_{\eta} M_{\sigma_{\eta}} \times \prod_q \mathbb{N} \right) \bigg/ R \right)^{sat}
\]

It is quite a bit more natural to describe the dual lattice

\[
Q_f^\vee = \left\{ \left( (v_{\eta}), (e_q) \right) \in \prod_{\eta} N_{\sigma_{\eta}} \times \prod_q \mathbb{N} \ \bigg| \ \forall \begin{array}{c} \eta_q^1 \rightarrow q \\ \eta_q^2 \end{array} \begin{array}{c} v_{q}^1 - v_{q}^2 = e_q u_q \end{array} \right\}.
\]
Tropical interpretation

Given a map $f$ over an $\mathbb{N}$-point we have a graph in $\Sigma(X)$ with
- vertices $v_i \in N_{\sigma \eta} \subset \sigma \eta$
- edges proportional to $u_q \in N^\text{gp}_q$ such that $v^1_q - v^2_q = e_q u_q$

this means
- The equations $v^1_q - v^2_q = e_q u_q$ define the cone of all such graphs
- $Q^\vee_f$ is the integer lattice in that cone.

Theorem (Gross-Siebert)

The minimal object exists, with characteristic sheaf $Q_f$, dual to the lattice in the corresponding space of tropical curves.