

Logarithmic Geometry and Moduli

Dan Abramovich, Qile Chen, Yuhao Huang, Martin Olsson,
Matt Satriano, and Shenghao Sun

ABSTRACT. We discuss the role played by logarithmic structures in the theory of moduli.

1. Introduction

intro

1.1.

It can be said that Logarithmic Geometry is concerned with a method of finding and using “hidden smoothness” in singular varieties. The original insight comes from consideration of de Rham cohomology. Since singular varieties naturally occur “at the boundary” of many moduli problems, Logarithmic Geometry was soon applied in the theory of moduli.

The main body of work on logarithmic geometry has been concerned with deep applications in the cohomological study of p -adic and arithmetic schemes. This gave the theory an aura of “yet another extremely complicated theory”, in the same general baggage as Fontaine’s big rings. It is a bit unfortunate that the founders were not inclined to pursue simple geometric applications right away to scatter this aura. The treatments of the theory are however quite accessible. We hope to convince the reader here that the theory is simple enough and useful enough to be considered by anybody interested in moduli of singular varieties, indeed enough to be included in a Handbook of Moduli.

1.2. Normal crossings and logarithmic smoothness

So what is the original insight? Let X be a nonsingular complex variety, S a curve with a point s and $f : X \rightarrow Y$ a dominant morphism smooth away from s , in such a way that $f^{-1}s = X_s = Y_1 \cup \dots \cup Y_m$ is a reduced simple normal crossings divisor. Then of course $\Omega_{X/Y} = \Omega_X/f^*\Omega_Y$ fails to be locally free at the singular points of f . But consider instead the sheaves $\Omega_X(\log(X_s))$ of differential forms with at most logarithmic poles along the Y_i , and similarly $\Omega_S(\log(s))$. Then there is an injective sheaf homomorphism $f^*\Omega_S(\log(s)) \rightarrow \Omega_X(\log(X_s))$, and *the quotient sheaf Ω_f^{\log} is locally free.*

2000 *Mathematics Subject Classification.* Primary 14A20; Secondary 14Dxx.

Key words and phrases. moduli, logarithmic structures.

So in terms of logarithmic forms, *the morphism f is as good as a smooth morphism.*

There is much more to be said: first, this Ω_f^{\log} can be extended to a logarithmic de Rham complex, and its hypercohomology, while not recovering the cohomology of the singular fibers, does give rise to the limiting Hodge structure. So it is evidently worth considering.

Second, the picture is quite a bit more general, and can be applied to all toric and toroidal maps between toric varieties or toroidal embeddings (with a little caveat about the characteristic of the residue fields). So there is some flexibility in choosing $X \rightarrow S$.

1.3. The search for a structure

Since we are considering moduli, then as soon as we consider $X \rightarrow S$ as above we must also consider the normal crossings fiber $X_s \rightarrow \{s\}$. But what structure should we put on this variety? The notion of differentials with logarithmic poles along X_s is not in itself intrinsic to X_s . Also the normal crossings variety X_s is not in itself toric or toroidal, so a new structure is needed to incorporate it into the picture.

One is tempted to consider varieties which are assembled from nice variety by some sort of gluing, as normal crossings varieties are. But already normal crossings varieties do not give a satisfactory answer in general, because their deformation spaces have “bad” components. Here is a classical example: consider a smooth projective variety Z such that $Pic^0(Z)$ is nontrivial. Let L be a line bundle on Z and set $Y = \mathbb{P}(\mathcal{O} \oplus L)$, with zero section $Z \subset Y$. Let X be the blowing up of $Z \times 0 \subset Y \times \mathbb{A}^1$. We have a flat morphism $f : X \rightarrow \mathbb{A}^1$ with fiber $X_0 = f^{-1}(0) \simeq Y \cup Y$, where the two copies of Y are glued with the zero section of one attached to the ∞ section of the other.

So clearly X_0 is a normal crossings variety with a nice smoothing to a copy of Y . But there are other deformations: the variety $Y \cup Y$ also deforms to $Y \cup Y'$ where $Y' = \mathbb{P}(\mathcal{O} \oplus L')$ and L' a deformation of the line bundle L . And it is not hard to see that $Y \cup Y'$ does not have a smoothing. Ideally one really does not want to see this deformation $Y \cup Y'$ in the picture - and ideally X_0 should have a natural structure whose deformation space excludes $Y \cup Y'$ automatically.

Such a structure was proposed by Friedman in [Fri83], where the notion of D -semistable varieties was introduced. This structure is somewhat subtle, and while it solves the issue in this case, it is not quite as flexible as one could wish. As we will see in Section 5, logarithmic structures subsume D -semistability and do provide an appropriate flexibility.

The purpose of this chapter is to describe logarithmic structure and to indicate where they can be useful in the study of moduli spaces.

1.4. This chapter is organized as follows

Section 2 gives the basic definitions of logarithmic structures, and section 3 discusses logarithmic differentials and log smooth deformations. Section 4 gives the first example where logarithmic geometry fits well with moduli spaces: the moduli space of stable curves is the moduli space of log smooth curves. The issue of D -semistability does not arise since a nodal curve is automatically D -semistable. So the theory for curves is simple. going towards higher dimensions, Section 5 shows how D -semistability can be described within logarithmic structures.

If one is to enlarge algebraic geometry to include logarithmic structures, the task of generalizing the techniques of algebraic geometry to logarithmic structure can certainly seem daunting. In section 6 we show how to encode logarithmic structure in terms of certain algebraic stacks. This allows us to reduce various constructions to the case of algebraic stacks. (One can argue that the theory of stacks is not simple either, but at least in the theory of moduli they have come to be accepted.)

In section 7 we make use of logarithmic stacks to describe the complexes which govern deformations and obstructions for logarithmic structures even in the non-smooth case. This comes in handy later. For instance, even when studying moduli of log-smooth schemes, the moduli spaces tend to be singular, and their cotangent complexes are a necessary ingredients in constructing virtual fundamental classes.

Section 8 describes a beautiful construction, similar to polar coordinates, in which families of complex log smooth varieties give rise canonically to families of topological manifolds. Differential geometers have used polar coordinates on nodal curves to “make space” for monodromy to act by Dehn twists. Rounding (using Ogus’s terminology) is a magnificent way to generalize this.

The immediate implications of logarithmic structures for De Rham cohomology and Hodge structures is described in Section 9.

We conclude by describing three applications, where logarithmic structures serve as the proverbial “magic powder” (term suggested by Kato and Ogus) to clarify or remove unwanted behavior from moduli spaces. Section 10 describes a number of cases where the main irreducible component of a moduli space can be separated from other “unwanted” components by sprinkling the objects with a bit of logarithmic structures. Section 11 gives a description of root constructions in terms of logarithmic structures. In particular it gives a palatable way to construct the moduli stack of twisted prestable curves. Section 12 describes work of B. Kim, in which Jun Li’s moduli space of relative stable maps, with its obstruction theory and virtual fundamental class, is beautifully simplified using logarithmic structures.

2. Definitions and basic properties

Qile1

In the first two sections, we will introduce the basic definitions of logarithmic geometry in the sense of [Kat89]. A good introduction would be [Kat89] and [Ogu01].

2.1. Logarithmic Structure

Definition 2.1. *A monoid is a commutative semi-group with a unit. A morphism of monoids is required to preserve the unit element. We use Mon to denote the category of Monoids.*

Definition 2.2. *Let X be a scheme. A pre-logarithmic structure on X is a sheaf of monoids \mathcal{M} on the étale site $X_{\text{ét}}$ combined with a morphism of sheaves of monoids: $\alpha : \mathcal{M} \rightarrow \mathcal{O}_X$, called the structure morphism, where we view \mathcal{O}_X as a monoid under multiplication. A pre-log structure is called a log structure if $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$ via α . We call the pair (X, \mathcal{M}) a log scheme. We use LSch to denote the category of log schemes.*

Definition 2.3. *Given two pre-log structures \mathcal{M} and \mathcal{N} on X . A morphism between them is a morphism $\mathcal{M} \rightarrow \mathcal{N}$ of sheaves of monoids which is compatible with the structure morphisms.*

2.2. Associated Log Structure

Given a pre-log structure $\alpha : \mathcal{M} \rightarrow \mathcal{O}_X$ on X , we can associate a log structure \mathcal{M}^a to be the push-out of

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^*) & \longrightarrow & \mathcal{M} \\ \downarrow & & \\ \mathcal{O}_X^* & & \end{array}$$

in the category of sheaves of monoids on $X_{\text{ét}}$, endowed with

$$\mathcal{M}^a \rightarrow \mathcal{O}_X \quad (a, b) \mapsto \alpha(a)b \quad (a \in \mathcal{M}, b \in \mathcal{O}_X^*).$$

In this way, we obtain a functor $a : (\text{pre-log structures on } X) \rightarrow (\text{log structures on } X)$. From the universal property of push-out, any morphism of pre-log structure from a pre-log structure \mathcal{M} to a log structure on X factor through \mathcal{M}^a uniquely.

On the other hand we have a natural inclusion $i : (\text{log structure on } X) \hookrightarrow (\text{pre-log structure on } X)$ by viewing log structure as a pre-log structure. So we have the following lemma.

Lemma 2.4. [Ogu01, 1.1.5] *The functor a is left adjoint to i .*

Example 2.5. *The category of log structures on X has an initial object, called the trivial log structure, given by the inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$. And it also has a final object, given by the identity map $\mathcal{O}_X \rightarrow \mathcal{O}_X$.*

NClog

Example 2.6. *Let X be a regular scheme, $D \subset X$ is a normal crossing divisor. We can define a log structure \mathcal{M} on X associated to the divisor D as $\mathcal{M} = \{g \in \mathcal{O} : g \text{ is invertible outside } D\} \subset \mathcal{O}_X$. Note that the concept of normal crossing is local in the étale topology. This is one of the reasons we use the étale topology instead of the Zariski topology.*

Afflog

Example 2.7. *Let P be a monoid, R a ring. Denote $X = \text{Spec} R[P]$, then X has a canonical log structure associated to the canonical map $P \rightarrow R[P]$. We usually use $\text{Spec}(P \rightarrow R[P])$ to denote the log scheme X with its canonical log structure.*

2.3. The inverse image

Let $f : X \rightarrow Y$ be a morphism of schemes. Given a log structure \mathcal{M}_Y on Y , we can obtain a log structure on X called the inverse image of \mathcal{M}_Y , to be the log structure associated to the pre-log structure $f^{-1}(\mathcal{M}_Y) \rightarrow f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$. This is usually denoted by $f^*(\mathcal{M}_Y)$. Using the inverse image of log structures, we can give the following definition.

Definition 2.8. *A morphism of log schemes $(X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ consists of a morphism of underlying schemes $f : X \rightarrow Y$, and a morphism $f^\flat : f^*\mathcal{M}_Y \rightarrow \mathcal{M}_X$ of log structures on X .*

Example 2.9. *In the example 2.7, the log structure on $\text{Spec}(P \rightarrow R[P])$ can be viewed as the inverse image of the log structure on $\text{Spec}(P \rightarrow \mathbb{Z}[P])$ via the canonical map $\text{Spec}(R[P]) \rightarrow \text{Spec}(\mathbb{Z}[P])$.*

Logpt

Example 2.10. *Let k be a field, $Y = \text{Spec } k[x_1, \dots, x_n]$, $D = V(x_1 \cdots x_r)$. Note that D is a normal crossing divisor in Y . By example 2.6, we have a log structure \mathcal{M}_Y on Y associated to the divisor D . In fact, \mathcal{M}_Y can be viewed as a subsheaf of \mathcal{O}_Y generated by \mathcal{O}_Y^* and $\{x_1, \dots, x_r\}$.*

Consider the inclusion $j : p = \text{Spec } k \hookrightarrow Y$ sending the point to the origin of Y . Then $j^\mathcal{M}_Y = k^* \oplus \mathbb{N}^r$, and the structure map $j^*\mathcal{M} \rightarrow \mathcal{O}_X$ is given by $(a, n_1, \dots, n_r) \mapsto a \cdot 0^{n_1 + \dots + n_r}$, where we assume $0^0 = 1$ and $0^n = 0$ if $n \neq 0$. Such point with the log structure above is called a logarithmic point, when $r = 1$ we call it the standard logarithmic point.*

2.4. Charts of log structures

Definition 2.11. *Let (X, \mathcal{M}_X) be a log scheme, and P a monoid. A chart of \mathcal{M}_X is a morphism $P \rightarrow \Gamma(X, \mathcal{M}_X)$, such that the associated log structure to the pre-log structure $P \rightarrow \Gamma(X, \mathcal{M}_X) \rightarrow \Gamma(X, \mathcal{O}_X)$ is \mathcal{M}_X .*

In fact, a chart of \mathcal{M}_X is equivalent to a morphism $f : (X, \mathcal{M}_X) \rightarrow \text{Spec}(P \rightarrow \mathbb{Z}[P])$, such that f^\flat is an isomorphism. In general, we have the following:

Lemma 2.12. [Ogu01, 1.1.9]

$$\mathrm{Hom}_{\mathrm{LSch}}((X, \mathcal{M}_X), \mathrm{Spec}(P \rightarrow \mathbb{Z}[P])) \cong \mathrm{Hom}_{\mathrm{Mon}}(P, \Gamma(X, \mathcal{M}_X)).$$

We can also consider the charts of log morphisms.

Definition 2.13. Let $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a morphism of log schemes. A chart of f is a triple $(P_X \rightarrow \mathcal{M}_X, Q_Y \rightarrow \mathcal{M}_Y, Q \rightarrow P)$ where P_X and Q_Y are the constant sheaves associated to the monoids P and Q , which satisfy the following conditions:

- (1) $P_X \rightarrow \mathcal{M}_X$ and $Q_Y \rightarrow \mathcal{M}_Y$ are charts of \mathcal{M}_X and \mathcal{M}_Y ;
- (2) the morphism of monoids $Q \rightarrow P$ makes the following diagram commutative:

$$\begin{array}{ccc} Q_X & \longrightarrow & P_X \\ \downarrow & & \downarrow \\ f^* \mathcal{M}_Y & \longrightarrow & \mathcal{M}_X. \end{array}$$

2.5. Fine log structures

In general, log structures are too wild to manipulate. Next we will introduce some well-behaved log structures. Given a monoid P , we can associate a group

$$P^{gp} := \{(a, b) \mid (a, b) \sim (c, d) \text{ if } \exists s \in P \text{ such that } s + a + d = s + b + c\}.$$

Note that any morphism from P to an abelian group factor through P^{gp} uniquely.

Definition 2.14. P is called *integral* if $P \rightarrow P^{gp}$ is injective. It is called *saturated* if it is integral and for any $p \in P^{gp}$, if $n \cdot p \in P$ for some positive integer n then $p \in P$.

Definition 2.15. A log scheme (X, \mathcal{M}_X) is said to be *fine*, if étale locally there is a chart $P \rightarrow \mathcal{M}$ with P a finitely generated integral monoid. If moreover P is saturated, (X, \mathcal{M}_X) is called *fine and saturated*, and we denote it *fs*.

In the following, we will focus on fine log schemes.

3. Differentials, smoothness, and log smooth deformations

Qile2

3.1. Logarithmic differentials

In [Gro64] where Grothendieck defines derivation using the difference of lifting sections. In our log case, we can do the same thing. First, we need a concept of infinitesimal extension, which requires the following definition.

Definition 3.1. A morphism $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ is called *strict* if $f^\flat : f^* \mathcal{M}_Y \rightarrow \mathcal{M}_X$ is an isomorphism. It is called a *strict closed immersion*¹ if it is strict and the underlying map $X \rightarrow Y$ is a closed immersion in the usual sense.

Let us consider the following commutative diagram:

$$\begin{array}{ccc} (T_0, \mathcal{M}_{T_0}) & \xrightarrow{\phi} & (X, \mathcal{M}_X) \\ J \downarrow j & \nearrow g_1 & \downarrow f \\ (T_1, \mathcal{M}_{T_1}) & \xrightarrow{\psi} & (Y, \mathcal{M}_Y) \end{array}$$

where j is an exact closed immersion defined by J with $J^2 = 0$. Note that T_0 and T_1 have the same underlying topological space. Then we have the following commutative diagram of sheaves of algebras given by the underlying maps:

$$\begin{array}{ccc} \mathcal{O}_{T_0} & \longleftarrow & \phi^{-1} \mathcal{O}_X \\ \uparrow & \nearrow g_1^\# & \uparrow \\ \mathcal{O}_{T_1} & \longleftarrow & \psi^{-1} \mathcal{O}_Y \end{array}$$

Then $g_1^\# - g_2^\#$ is a derivation $\partial_{g_1 - g_2} : \phi^{-1} \mathcal{O}_X \rightarrow J$ in the usual sense. We also have a commutative diagram given by the log structures:

$$\begin{array}{ccc} \mathcal{M}_{T_0} & \longleftarrow & \phi^{-1} \mathcal{M}_X \\ \uparrow & \nearrow g_1^\flat & \uparrow \\ \mathcal{M}_{T_1} & \longleftarrow & \psi^{-1} \mathcal{M}_Y \end{array}$$

Note that we have an exact sequence of multiplicative monoids

$$\mathbf{1} \rightarrow (1 + J) \rightarrow \mathcal{M}_{T_1} \rightarrow \mathcal{M}_{T_0} \rightarrow \mathbf{1}.$$

Hence we obtain a morphism $D_{g_1 - g_2} : \phi^{-1} \mathcal{M}_X \rightarrow J$ such that $\forall m \in \phi^{-1} \mathcal{M}_X$, we have $(g_1^\flat - g_2^\flat)(m) = 1 + D_{g_1 - g_2} m$. Since J is now viewed as an additive group, we obtain $D_{g_1 - g_2} : \phi^{-1} \mathcal{M}_X^{gp} \rightarrow J$, and it is not hard to check that $D_{g_1 - g_2}(m \cdot n) = D_{g_1 - g_2}(m) + D_{g_1 - g_2}(n)$ for any $m, n \in \phi^{-1}(\mathcal{M}_X)$. By the definition of log structures, we also have

- (1) $\alpha(m) D_{g_1 - g_2} m = \partial_{g_1 - g_2}(\alpha(m)), \forall m \in \phi^{-1} \mathcal{M}_X$;
- (2) $D_{g_1 - g_2}|_{\psi^{-1} \mathcal{M}_Y} = 0$.

Remark 3.2. (1) *Since the log structure contains all the invertible elements in the structure sheaf, the map $D_{g_1 - g_2}$ determines $\partial_{g_1 - g_2}$.*

¹In [Kat89], this is called an exact closed immersion. But I feel more explicit if using the word “strict” instead of “exact”.

- (2) The above properties show that $D_{g_1-g_2}$ behaves like "dlog". This is one of the reasons for the name "logarithmic structure".

Summarizing the above discussion gives the following definitions:

LogDer

Definition 3.3. Let $\bar{X} = (X, \mathcal{M}_X)$, $\bar{Y} = (Y, \mathcal{M}_Y)$ be fine log schemes, and $f : \bar{X} \rightarrow \bar{Y}$ a morphism of log schemes. Let I be an \mathcal{O}_X -module. The sheaf $Der_{\bar{Y}}(\bar{X}, I)$ of log derivations of \bar{X} over \bar{Y} to I is the sheaf of germs of pairs (∂, D) where $\partial \in Der_Y(X, I)$ and $D : \mathcal{M}_X \rightarrow I$ such that the following conditions hold:

- (1) $D(ab) = D(a) + D(b)$ for $a, b \in \mathcal{M}_X$;
- (2) $\alpha(a)D(a) = \partial(\alpha(a))$, for $a \in \mathcal{M}_X$.
- (3) $D(a) = 0$, for $a \in f^{-1}\mathcal{M}_Y$.

LodDiff

Definition 3.4. Using the notation as above, we define the \mathcal{O}_X -module Ω_f^{log} to be the quotient $\Omega_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{gp})/\mathcal{K}$, where \mathcal{K} is a \mathcal{O}_X -module generated by local sections of the following forms:

- (1) $(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$ with $a \in \mathcal{M}_X$;
- (2) $(0, 1 \otimes a)$ with $a \in \text{Im}(f^{-1}(\mathcal{M}_Y) \rightarrow \mathcal{M}_X)$.

The sheaf Ω_f^{log} is called the sheaf of logarithmic differentials.

Remark 3.5. If we consider only fine log structures, and assume that Y is locally noetherian and X locally of finite type over Y , then $Der_{\bar{Y}}(\bar{X}, I)$ and Ω_f^{log} in the definitions above are coherent sheaves. The proof of this can be found in [Ogu01, IV.1.1]

We have the following universal property relating the two definitions:

Proposition 3.6. [Ogu01, IV.1.1.6] Using the notations above, we have a natural isomorphism

$$\text{Hom}_{\mathcal{O}_X}(\Omega_f^{log}, I) \cong Der_{\bar{Y}}(\bar{X}, I),$$

given by $u \mapsto (u \circ \partial, u \circ D)$, where (∂, D) are the universal derivation defined by $\partial : \mathcal{O}_X \rightarrow \Omega_{X/Y} \rightarrow \Omega_f^{log}$ and $D : \mathcal{M}_X \rightarrow \mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{gp} \rightarrow \Omega_f^{log}$.

NCsmooth

Example 3.7. Let $R = k[x_1, \dots, x_n]/(x_1 \cdots x_r)$, where k is a field. Denote $X = \text{Spec}R$, and \mathcal{M}_X to be the log structure on X given by $\mathbb{N}^r \rightarrow R$, $e_i \mapsto x_i$, where e_i is the standard generator of the monoid \mathbb{N}^r . Let $(Y, \mathcal{M}_Y) = \text{Spec}(\mathbb{N} \rightarrow k)$ be the logarithmic point described in 2.10. Now we can define a morphism $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ by the following diagram:

$$\begin{array}{ccc} \mathbb{N}^r & \longrightarrow & R \\ \Delta \uparrow & & \uparrow \\ \mathbb{N} & \longrightarrow & k \end{array}$$

where $\Delta : e \mapsto e_1 + \cdots + e_r$, and e is the standard generator of \mathbb{N} . Then it is easy to see that $\text{Der}_{(Y, \mathcal{M}_Y)}((X, \mathcal{M}_X), \mathcal{O}_X)$ is a free \mathcal{O}_X -module generated by $x_1 \frac{\partial}{\partial x_1}, \dots, x_r \frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_{r+1}}, \dots, \frac{\partial}{\partial x_n}$, with a relation $x_1 \frac{\partial}{\partial x_1} + \cdots + x_r \frac{\partial}{\partial x_r} = 0$. The sheaf Ω_f^{\log} is a free \mathcal{O}_X -module generated by the logarithmic differentials: $\frac{dx_1}{x_1}, \dots, \frac{dx_r}{x_r}, x_{r+1}, \dots, x_n$, with a relation $\frac{dx_1}{x_1} + \cdots + \frac{dx_r}{x_r} = 0$.

Example 3.8. Let $h : Q \rightarrow P$ be a morphism of fine monoids. Denote $(X, \mathcal{M}_X) = \text{Spec}(P \rightarrow \mathbb{Z}[P])$ and $(Y, \mathcal{M}_Y) = \text{Spec}(Q \rightarrow \mathbb{Z}[Q])$. Then we have a morphism $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ induced by h . A direct calculation shows that $\Omega_f^{\log} = \mathcal{O}_X \otimes \text{Cok}(h^{gp})$.

3.2. Logarithmic Smoothness

Let us go back to the following diagram:

$$\begin{array}{ccc} (T_0, \mathcal{M}_{T_0}) & \xrightarrow{\phi} & (X, \mathcal{M}_X) \\ J \downarrow j & & \downarrow f \\ (T_1, \mathcal{M}_{T_1}) & \xrightarrow{\psi} & (Y, \mathcal{M}_Y) \end{array}$$

where j is an exact closed immersion defined by J with $J^2 = 0$. As in the usual case, we can define log smoothness by the existence of lifting sections.

Definition 3.9. A morphism $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ of fine log schemes is called smooth (étale) if the underlying morphism $X \rightarrow Y$ is locally of finite presentation and for any commutative diagram as above, there exists étale locally on T_1 a (unique) morphism $g : (T_1, \mathcal{M}_{T_1}) \rightarrow (X, \mathcal{M}_X)$ such that $\phi = g \circ j$ and $\psi = f \circ g$.

We have the following useful criterion for smoothness from [Kat89, Theorem 3.5].

KatoStrThm

Theorem 3.10. (K.Kato) Let $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ be a morphism of fine log schemes. Assume we have a chart $Q \rightarrow \mathcal{M}_Y$, where Q is a finitely generated integral monoid. Then the following are equivalent:

- (1) f is log smooth (étale);
- (2) étale locally on X , there exists a chart $(P_X \rightarrow \mathcal{M}_X, Q_Y \rightarrow \mathcal{M}_Y, Q \rightarrow P)$ extending the chart $Q_Y \rightarrow \mathcal{M}_Y$, satisfying the following properties.
 - (a) The kernel and the torsion part of the cokernel (the kernel and the cokernel) of $Q^{gp} \rightarrow P^{gp}$ are finite groups of order invertible on X .
 - (b) The induced morphism from $X \rightarrow Y \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P]$ is étale in the classical sense.

Remark 3.11. (1) We can require $Q^{gp} \rightarrow P^{gp}$ in (a) to be injective, and replace the étaleness of $X \rightarrow Y \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P]$ in (b) by the smoothness without changing the conclusion of the theorem 3.10.

- (2) *The arrow in (b) shows that log smooth arrow is "locally toric" relative to the base. If we consider the case $Y = \text{Spec } \mathbb{C}$ with the trivial log structure, and $X = \text{Spec}(P \rightarrow \mathbb{C}[P])$ where P is a fine, saturated and torsion free monoid. Then X is a toric variety with the action of $\text{Spec } \mathbb{C}[P^{gp}]$. According to the theorem, X is log smooth relative to Y , though the underlying space might be singular. These singularities are called toric singularities in the sense of [Kat94].*

Example 3.12. *Using the theorem, we can check directly that the morphism f in example 3.7 is log smooth, but the underlying map has normal crossing singularities. We will see later that one of the major advantages of log structure is to deal with the normal crossing singularities.*

For log smooth morphisms, the following proposition shows that log differentials behave like the usual differentials of smooth morphisms.

Proposition 3.13. *Let $(X, \mathcal{M}_X) \xrightarrow{f} (Y, \mathcal{M}_Y) \xrightarrow{g} (Z, \mathcal{M}_Z)$ be morphisms of fine log schemes.*

- (1) *There exists an exact sequence $f^*\Omega_g^{log} \rightarrow \Omega_{g \circ f}^{log} \rightarrow \Omega_f^{log} \rightarrow 0$.*
- (2) *If f is log smooth, then Ω_f^{log} is a locally free \mathcal{O}_X -module, and we have the following exact sequence: $0 \rightarrow f^*\Omega_g^{log} \rightarrow \Omega_{g \circ f}^{log} \rightarrow \Omega_f^{log} \rightarrow 0$.*
- (3) *If $g \circ f$ is log smooth and the sequence in (2) is exact and splits locally, then f is log smooth.*

A proof can be found in [Ogu01, Chapter IV].

3.3. Logarithmic smooth deformation

After we have the log smoothness, a natural thing to do is to develop the log smooth deformation. In many cases, we would require this to be a flat deformation for the underlying space. Unfortunately log smoothness does not imply flatness, so we need the following definition.

Definition 3.14. *A map of fine monoids $h : Q \rightarrow P$ is called integral if the induced map on monoid algebra $\mathbb{Z}[Q] \rightarrow \mathbb{Z}[P]$ is flat.*

Definition 3.15. *A morphism $f : (X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ of integral log schemes is called integral if for every geometric point $\bar{x} \in X$, the map of monoids $h : f^{-1}(\mathcal{M}_Y/\mathcal{O}_{Y,\bar{x}}^*) \rightarrow (\mathcal{M}_X/\mathcal{O}_{X,\bar{x}}^*)$ is integral.*

- Remark 3.16.**
- (1) *If f is integral, then étale locally we have a chart $(P_X \rightarrow \mathcal{M}_X, Q_Y \rightarrow \mathcal{M}_Y, Q \xrightarrow{h} P)$ such that h is integral.*
 - (2) *If $h : Q \rightarrow P$ is integral of integral monoids, then for any integral monoid Q' , the push-out of $P \leftarrow Q \rightarrow Q'$ in the category of monoids is integral. Thus integral morphisms are stable under base change by integral log schemes.*

- (3) Given a morphism $h : Q \rightarrow P$ of integral monoids, there is an explicit criterion, which looks complicated, but sometimes useful for checking integrality of h directly: if $a_1, a_2 \in Q$, $b_1, b_2 \in P$ and $h(a_1)b_1 = h(a_2)b_2$, then there exist $a_3, a_4 \in Q$ and $b \in P$ such that $b_1 = h(a_3)b$, $b_2 = h(a_4)b$ and $a_1a_3 = a_2a_4$.

Now we have the following fact from [Kat89, 4.5].

Proposition 3.17. *If f is a log smooth and integral morphism of fine log schemes, then \underline{f} the underlying map is flat in the usually sense.*

Now let us consider the following deformation problem. Given a log smooth integral morphism $f_0 : (X_0, \mathcal{M}_{X_0}) \rightarrow (B_0, \mathcal{M}_{B_0})$ of fine log schemes, and a exact closed immersion $j : (B_0, \mathcal{M}_{B_0}) \rightarrow (B, \mathcal{M}_B)$ defined by an ideal J with $J^2 = 0$. We want to find a log smooth lifting $f : (X, \mathcal{M}_X) \rightarrow (B, \mathcal{M}_B)$ fits into the following cartesian diagram:

$$\begin{array}{ccc} (X_0, \mathcal{M}_{X_0}) & \hookrightarrow & (X, \mathcal{M}_X) \\ \downarrow & & \downarrow \\ (B_0, \mathcal{M}_{B_0}) & \hookrightarrow & (B, \mathcal{M}_B). \end{array}$$

Remark 3.18. *Since f_0 is integral, and $\mathcal{M}_X/(1+J) \cong \mathcal{M}_{X_0}$, it is not hard to show that the lifting f is automatically integral and hence flat.*

We denote $T_{f_0}^{log} = (\Omega_{f_0}^{log})^\vee$ to be the log tangent sheaf for any morphism f_0 of fine log schemes. We have the following theorem for log smooth deformations.

Theorem 3.19. *With the notation as above, we have:*

- (1) *There is a canonical obstruction $\eta \in H^2(X_0, T_{f_0}^{log} \otimes J)$ such that $\eta = 0$ if and only if there is an log smooth lifting.*
- (2) *If $\eta = 0$, then the set of log smooth deformations form a torsor under $H^1(X_0, T_{f_0}^{log} \otimes J)$.*
- (3) *The automorphism of any deformation is isomorphic to $H^0(X_0, T_{f_0}^{log} \otimes J)$.*

The theorem can be proved similar to the case of usual deformation theory. Another proof using logarithmic cotangent complex can be found in [Ols05, Thm 5.6], which we will discuss later.

4. Log smooth curves and their moduli

Satriano

5. D-semistability and log structures

D-SS

6. Stacks of logarithmic structures

LogStacks

Before introducing the stack LOG_S classifying fine log structures on schemes over a fine log scheme S , constructed by Olsson in [Ols03], let's look at an example, which will give the local covers of LOG_S .

For a fine log scheme X with log structure $(\mathcal{M}_X, \alpha : \mathcal{M}_X \rightarrow \mathcal{O}_X)$, we identify \mathcal{O}_X^* with its inverse image under α , regarded as a subsheaf of monoids in \mathcal{M}_X . Let $\overline{\mathcal{M}}_X$ be the quotient sheaf $\mathcal{M}_X/\mathcal{O}_X^*$ for the étale topology, called the characteristic of X . We write \underline{X} for the underlying scheme of the log scheme X . See [Kat89].

Define the *Deligne-Faltings log structure* of rank r on \underline{X} to be a family (L_1, \dots, L_r) of line bundles on \underline{X} together with maps $s_i : L_i \rightarrow \mathcal{O}_X$ of line bundles, for each i .

Consider the following three functors from schemes to groupoids:

- ① $X \mapsto \{\text{Deligne-Faltings log structures of rank 1 on } X\}$;
- ② $X \mapsto \{\text{fine log structures } \mathcal{M}_X \xrightarrow{\alpha} \mathcal{O}_X \text{ with a morphism of sheaves of monoids } \mathbb{N} \xrightarrow{\beta} \overline{\mathcal{M}}_X \text{ that étale locally lifts to a chart: } \mathbb{N} \xrightarrow{\tilde{\beta}} \mathcal{M}_X\}$;
- ③ $X \mapsto [\mathbb{A}^1/\mathbb{G}_m](X)$, where the quotient stack is formed with respect to the multiplication action of \mathbb{G}_m on \mathbb{A}^1 .

Lemma 6.1. *These three functors are equivalent.*

Let's sketch the proof. Given a DF log structure $(L, s : L \rightarrow \mathcal{O}_X)$ of rank 1 on X , define a sheaf of monoids \mathcal{M}' on X to be

$$\coprod_{n \geq 0} \underline{\text{Isom}}(\mathcal{O}_X, L^{\otimes n}),$$

the sheafification of the presheaf that takes U to $\coprod_{n \geq 0} \underline{\text{Isom}}(\mathcal{O}_U, (L|_U)^{\otimes n})$. It comes with a natural morphism of sheaves of monoids $\mathcal{M}' \rightarrow \mathbb{N}$. The monoid structure on \mathcal{M}' is the obvious one:

$$(n, a : \mathcal{O} \rightarrow L^{\otimes n}) \cdot (m, b : \mathcal{O} \rightarrow L^{\otimes m}) = (n + m, a \otimes b).$$

The map $s : L \rightarrow \mathcal{O}_X$ induces a morphism

$$\underline{\text{Isom}}(\mathcal{O}_X, L^{\otimes n}) \xrightarrow{\otimes s} \underline{\text{Hom}}(\mathcal{O}_X, \mathcal{O}_X) = \mathcal{O}_X$$

of sheaves, hence giving a pre-log structure on \mathcal{M}' :

$$\mathcal{M}' \rightarrow \mathcal{O}_X.$$

We take \mathcal{M}_X to be the log structure associated to this pre-log structure \mathcal{M}' . Note that $\mathcal{M}'/\mathcal{O}_X^* \cong \mathbb{N}$, and we define $\beta : \mathbb{N} \rightarrow \overline{\mathcal{M}}_X$ to be the composite

$$\beta : \mathbb{N} \cong \mathcal{M}'/\mathcal{O}_X^* \rightarrow \mathcal{M}/\mathcal{O}_X^*.$$

Locally the line bundle L is trivial, and one can choose a trivialization of L , which gives trivializations of all $L^{\otimes n}$. Sending $n \in \mathbb{N}$ to this trivialization defines a section $\mathbb{N} \rightarrow \mathcal{M}'$, and hence a section $\tilde{\beta} : \mathbb{N} \rightarrow \mathcal{M}' \rightarrow \mathcal{M}_X$. One can check that this is a chart.

Conversely, given a fine log structure $(\mathcal{M}_X, \alpha : \mathcal{M}_X \rightarrow \mathcal{O}_X)$ on X with a morphism $\beta : \mathbb{N} \rightarrow \overline{\mathcal{M}}_X$ that étale locally lifts to a chart $\tilde{\beta} : \mathbb{N} \rightarrow \mathcal{M}_X$, we have a

section $\beta(1)$ of $\overline{\mathcal{M}}_X$, and its inverse image under $\pi : \mathcal{M}_X \rightarrow \overline{\mathcal{M}}_X$ is an \mathcal{O}_X^* -torsor, which corresponds to a line bundle L . The composition

$$\pi^{-1}(\beta(1)) \subset \mathcal{M}_X \xrightarrow{\alpha} \mathcal{O}_X$$

gives a morphism of line bundles $s : L \rightarrow \mathcal{O}_X$.

Giving a morphism $X \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to giving a \mathbb{G}_m -torsor (namely a line bundle L) with a \mathbb{G}_m -equivariant morphism to \mathbb{A}^1 :

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

This diagram is equivalent to the following one

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{A}_X^1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]_X, \end{array}$$

and the top arrow induces a \mathbb{G}_m -equivariant morphism $Y \rightarrow \mathbb{A}_X^1$, namely a morphism of line bundles $s : L \rightarrow \mathcal{O}_X$. This finishes the proof.

In fact, in the three functors, one can replace \mathbb{N} by \mathbb{N}^r , and rank 1 DF-log structure by rank r DF-log structure, and replace $[\mathbb{A}^1/\mathbb{G}_m]$ by $[\mathbb{A}^r/\mathbb{G}_m^r]$, and they are still equivalent.

More generally, let P be a fine monoid and S a scheme, and let $S[P]$ be the product $S \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} \mathbb{Z}[P]$, which has a fine log structure coming from the chart $P \rightarrow \mathbb{Z}[P]$. Let P^{gp} be the associated group, then we have the following.

Lemma 6.2. *The following two functors from S -schemes to groupoids are equivalent:*

① $X \mapsto \{\text{fine log structures } \mathcal{M}_X \xrightarrow{\alpha} \mathcal{O}_X \text{ with a morphism of sheaves of monoids } P \xrightarrow{\beta} \overline{\mathcal{M}}_X \text{ that fppf locally lifts to a chart: } P \xrightarrow{\tilde{\beta}} \mathcal{M}_X\};$

② $X \mapsto [S[P]/S[P^{\mathrm{gp}}]](X).$

If in addition P is fs, then one can replace “fppf” by “étale”.

The S -group scheme $S[P^{\mathrm{gp}}]$ acts on $S[P]$ by translation. Note that for an affine S -scheme $\mathrm{Spec} R$, the set of R -points $S[P](R)$ is the set of monoid homomorphisms $\mathrm{Hom}_{\mathrm{mon}}(P, R)$, where R is regarded as a multiplicative monoid. When $S = \mathrm{Spec} k$ for a field k and P is saturated and torsion-free, the variety $S[P]$ with this action of the torus $S[P^{\mathrm{gp}}]$ is a toric variety, and the stack quotient $\mathcal{S}_P = [S[P]/S[P^{\mathrm{gp}}]]$ is a toric stack.

Now we can discuss the stack LOG_S parameterizing fine log structures.

Let S be a fine log scheme. Define LOG_S to be the category with

- objects: morphisms $(X, \mathcal{M}_X) \rightarrow (S, \mathcal{M}_S)$ of fine log schemes, and
- morphisms: strict morphisms $(X, \mathcal{M}_X) \rightarrow (Y, \mathcal{M}_Y)$ over S .

With the functor $(X \rightarrow S) \mapsto (\underline{X} \rightarrow \underline{S})$ from $LOG_S \rightarrow \text{Sch}_{\underline{S}}$, this defines a fibered category over \underline{S} . One of the main results in [Ols03] is the following.

Theorem 6.3. *LOG_S is an algebraic stack locally of finite presentation over \underline{S} .*

Here for an algebraic stack we use a slightly different definition from [GL00, 4.1]. Namely the first axiom there that the diagonal is representable, separated and quasi-compact, is replaced by that the diagonal is representable and of finite presentation. In fact the stack LOG_S is not quasi-separated [Ols03, 3.17].

Here are two basic properties of LOG_S .

Proposition 6.4. (1). *The natural map $i_S : \underline{S} \rightarrow LOG_S$ corresponding to the given fine log structure \mathcal{M}_S on S is an open immersion;*

(2). *The 2-functor*

$$S \mapsto LOG_S : \{\text{fine log schemes}\} \rightarrow \{\text{algebraic stacks}\}$$

preserves fiber product. More precisely, if

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

is a Cartesian square of fine log schemes, then the induced diagram

$$\begin{array}{ccc} LOG_{X'} & \longrightarrow & LOG_X \\ \downarrow & & \downarrow \\ LOG_{S'} & \longrightarrow & LOG_S \end{array}$$

is a 2-Cartesian square of algebraic stacks.

We don't give their proof here, which can be found in [Ols03].

One can use this stack LOG_S to reinterpret many concepts in log geometry. Note that for a morphism $f : X \rightarrow S$ of fine log schemes, the induced morphism $LOG(f) : LOG_X \rightarrow LOG_S$ is faithful, hence representable.

Definition 6.5. *Let P be a property of representable morphisms of algebraic stacks. Then we say that $f : X \rightarrow S$ has property $LOG(P)$ if $LOG(f) : LOG_X \rightarrow LOG_S$ has property P . We say that f has property weak $LOG(P)$ if the map $\underline{X} \rightarrow LOG_S$ corresponding to the given log structure \mathcal{M}_X has property P .*

Caution: The diagram

$$\begin{array}{ccc} \underline{X} & \xrightarrow{i_X} & LOG_X \\ \underline{f} \downarrow & & \downarrow LOG(f) \\ \underline{S} & \xrightarrow{i_S} & LOG_S \end{array}$$

does not necessarily commute. It commutes if and only if f is strict.

Recall that a morphism of fine log schemes $f : X \rightarrow S$ is said to be *formally log smooth* (resp. *formally log étale*) if in the following commutative diagram of fine log schemes, étale locally on T there exists (resp. exists a unique) a lifting

$$\begin{array}{ccc} T_0 & \longrightarrow & X \\ \downarrow i & \nearrow & \downarrow f \\ T & \longrightarrow & S \end{array}$$

for any strict square zero thickening $i : T_0 \rightarrow T$. We say that f is *log smooth* (resp. *log étale*) if it is formally log smooth (resp. formally log étale) and \underline{f} is locally of finite presentation. One defines (formal) smoothness and étaleness for representable morphisms of algebraic stacks in the similar way.

Theorem 6.6. *For a morphism $f : X \rightarrow S$ of fine log schemes, f is LOG smooth (resp. LOG étale) if and only if f is log smooth (resp. log étale), if and only if f is weakly LOG smooth (resp. weakly LOG étale).*

This is part of [Ols03, 4.6].

Another application is the following. Let $f_0 : X_0 \rightarrow T_0$ be a log smooth integral morphism of fine log schemes, and $T_0 \xrightarrow{i} T$ a strict square zero thickening defined by the ideal $I \subset \mathcal{O}_T$. Then we can consider the deformation problem of classifying Cartesian squares

$$\begin{array}{ccc} X_0 & \cdots \longrightarrow & X \\ f_0 \downarrow & & \downarrow f \\ T_0 & \xrightarrow{i} & T, \end{array}$$

where $f : X \rightarrow T$ is also log smooth (integral). Then the solution to this deformation problem is the cohomology of the *log tangent bundle* T_{X_0/T_0}^{\log} , namely

- there is an obstruction class $o \in H^2(\underline{X}_0, T_{X_0/T_0}^{\log} \otimes f_0^* I)$;
- when the obstruction class vanishes so that deformations exist, the set of all isomorphism classes of deformations is a torsor under $H^1(\underline{X}_0, T_{X_0/T_0}^{\log} \otimes f_0^* I)$;
- the automorphism group of each deformation is $H^0(\underline{X}_0, T_{X_0/T_0}^{\log} \otimes f_0^* I)$.

The morphism $f_0 : X_0 \rightarrow T_0$ of fine log schemes gives a smooth morphism $\underline{X}_0 \xrightarrow{f_0} LOG_{T_0}$, also denoted f_0 , and the above deformation problem is equivalent

to the following

$$\begin{array}{ccc} \underline{X}_0 & \cdots\cdots\cdots & \underline{X} \\ f_0 \downarrow & & \downarrow f \\ \text{LOG}_{T_0} & \longrightarrow & \text{LOG}_T. \end{array}$$

The solution to this deformation problem is the cohomology of the *ordinary tangent bundle* $T_{\underline{X}_0/\text{LOG}_{T_0}}$. In fact, we have $\Omega_{\underline{X}_0/T_0}^{\log} \cong \Omega_{\underline{X}_0/\text{LOG}_{T_0}}$, because they represent the same functor.

Finally, the stack LOG_S can be covered by toric stacks discussed above. Let S be a fine log scheme and $U \rightarrow S$ an étale map, and U is given the inverse image log structure. Let $Q \xrightarrow{\beta} \mathcal{M}_U$ be a chart, and $h : Q \rightarrow P$ a morphism of fine monoids, then we have a natural morphism $\mathcal{S}_P \times_{\mathcal{S}_Q} \underline{U} \rightarrow \text{LOG}_U$.

Proposition 6.7. *For any fine log scheme S , the natural morphism*

$$\coprod_{(U, \beta, h)} \mathcal{S}_P \times_{\mathcal{S}_Q} \underline{U} \rightarrow \text{LOG}_S$$

is a representable étale surjection, where the disjoint union is taken over all triples (U, β, h) consisting of an étale morphism $U \rightarrow S$, a chart $Q \xrightarrow{\beta} \mathcal{M}_U$, and a morphism $h : Q \rightarrow P$ of fine monoids, for some fine monoids P and Q .

7. Log deformation theory in general

LogDef

8. Rounding

Rounding

9. Log De Rham and hodge structures

DeRham

10. The main component of moduli spaces

MainComp

11. Log twisting and root constructions

Roots

12. Log stable maps

Stablemaps

References

Friedman

[Fri83] Robert Friedman, *Global smoothings of varieties with normal crossings*, Ann. of Math. (2) **118** (1983), no. 1, 75–114. MR MR707162 (85g:32029) 2

LMB

[GL00] Laurent Moret-Bailly Gérard Laumon, *Champs algébriques*, vol. 39, Springer-Verlag, 2000. 14

- EGA_IV** [Gro64] A. Grothendieck, *Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I*, no. 20, 1964. MR MR0173675 (30 #3885) [6](#)
- KKato** [Kat89] Kazuya Kato, *Logarithmic structures of Fontaine-Illusie*, 191–224. MR MR1463703 (99b:14020) [4](#), [7](#), [9](#), [11](#), [12](#)
- ToricSing** [Kat94] ———, *Toric singularities*, Amer. J. Math. **116** (1994), no. 5, 1073–1099. MR MR1296725 (95g:14056) [10](#)
- Ogus** [Ogu01] Arthur Ogus, *Lectures on logarithmic algebraic geometry*, TeXed notes, 2001. [4](#), [6](#), [8](#), [10](#)
- LogStack** [Ols03] Martin C. Olsson, *Logarithmic geometry and algebraic stacks*, Ann. Sci. d'ENS **36** (2003), 747–791. [11](#), [14](#), [15](#)
- LogCot** [Ols05] ———, *The logarithmic cotangent complex*, Math. Ann. **333** (2005), 859–931. [11](#)

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, BOX 1917, PROVIDENCE, RI 02912, U.S.A.

E-mail address: abrmovic@math.brown.edu

DEPARTMENT OF MATHEMATICS, BROWN UNIVERSITY, BOX 1917, PROVIDENCE, RI 02912, U.S.A.

E-mail address: q.chen@math.brown.edu

UNIVERSITY OF CALIFORNIA, BERKELEY

E-mail address: huang@math.berkeley.edu

E-mail address: molsson@math.berkeley.edu

UNIVERSITY OF CALIFORNIA, BERKELEY

E-mail address: satriano@math.berkeley.edu

UNIVERSITY OF CALIFORNIA, BERKELEY

E-mail address: shenghao@Math.Berkeley.EDU