

Logarithmic Geometry and Moduli

Dan Abramovich, Qile Chen, Danny Gillam, Yuhao Huang,
Martin Olsson, Matthew Satriano, and Shenghao Sun

ABSTRACT. We discuss the role played by logarithmic structures in the theory of moduli.

Contents

1	Introduction	2
1.1	Logarithmic structures in algebraic geometry	2
1.2	Normal crossings and logarithmic smoothness	3
1.3	The search for a structure	3
1.4	Organization of this chapter	4
1.5	Notation	5
1.6	Acknowledgements	5
2	Definitions and basic properties	6
2.1	Logarithmic structures	6
2.2	The log structure associated to a pre-log structure	6
2.3	The inverse image and the category of log schemes	7
2.4	Charts of log structures	8
2.5	Fine log structures	9
3	Differentials, smoothness, and log smooth deformations	9
3.1	Logarithmic differentials	9
3.2	Logarithmic Smoothness	12
3.3	Logarithmic smooth deformation	13
4	Log smooth curves and their moduli	15
4.1	Relative characteristic sheaves	15
4.2	Log curves	16
4.3	Log structures on stable curves	17
4.4	Moduli	18
4.5	Back to the big picture	19
5	D-semistability and log structures	20
5.1	Introduction	20

2000 *Mathematics Subject Classification.* Primary 14A20; Secondary 14Dxx.

Key words and phrases. moduli, logarithmic structures.

5.2 Refined analysis of the existence of log structures	23
6 Stacks of logarithmic structures	24
6.1 A motivating example	24
6.2 The stack of log structures	27
6.3 What is LOG_S good for?	28
6.4 Local structure of LOG_S .	29
7 Log deformation theory in general	30
7.1 The Log Cotangent Complex	30
7.2 Basic Properties	31
7.3 Deformation Theory of Log Schemes in General	33
7.4 Deformations of morphisms	34
8 Rounding	35
8.1 What is rounding?	35
8.2 The oriented real blowup	36
8.3 The Kato–Nakayama space	37
8.4 Relating Kato–Nakayama spaces to oriented blowups	37
8.5 Topology, cohomology, and the Kato–Nakayama space	38
8.6 Kato–Nakayama spaces of expanded pairs	39
9 Log De Rham and hodge structures	40
9.1 Moduli spaces of polarized Hodge structures.	40
9.2 Logarithmic Hodge structures.	42
9.3 Kato–Usui spaces.	44
10 The main component of moduli spaces	45
10.1 Moduli: compactness and main components	45
10.2 Example: the toric Hilbert scheme	46
11 Twisted curves and log twisted curves	49
11.1 Twisted curves	49
11.2 Log twisted curves	51
11.3 From twisted curves to log twisted curves	52
11.4 From log twisted curves to twisted curves	53
12 Log stable maps	54
12.1 From curves to maps and expansions	54
12.2 Logarithmic methods: from Jun Li to Bumsig Kim	56
12.3 Unexpanded log maps: from Siebert into the future	56

1. Introduction

intro

1.1. Logarithmic structures in algebraic geometry

It can be said that Logarithmic Geometry is concerned with a method of finding and using “hidden smoothness” in singular varieties. The original insight

comes from consideration of de Rham cohomology. Since singular varieties naturally occur “at the boundary” of many moduli problems, logarithmic geometry was soon applied in the theory of moduli.

The main body of work on logarithmic geometry has been concerned with deep applications in the cohomological study of p -adic and arithmetic schemes. This gave the theory an aura of “yet another extremely complicated theory”. The treatments of the theory are however quite accessible. We hope to convince the reader here that the theory is simple enough and useful enough to be considered by anybody interested in moduli of singular varieties, indeed enough to be included in a Handbook of Moduli.

1.2. Normal crossings and logarithmic smoothness

So what is the original insight? Let X be a nonsingular complex variety, S a curve with a point s and $f : X \rightarrow S$ a dominant morphism smooth away from s , in such a way that $f^{-1}s = X_s = Y_1 \cup \dots \cup Y_m$ is a reduced simple normal crossings divisor. Then of course $\Omega_{X/S} = \Omega_X/f^*\Omega_S$ fails to be locally free at the singular points of f . But consider instead the sheaves $\Omega_X(\log(X_s))$ of differential forms with at most logarithmic poles along the Y_i , and similarly $\Omega_S(\log(s))$. Then there is an injective sheaf homomorphism $f^*\Omega_S(\log(s)) \rightarrow \Omega_X(\log(X_s))$, and the quotient sheaf $\Omega_X(\log(X_s))/\Omega_S(\log(s))$ is locally free.

So in terms of logarithmic forms, the morphism f is as good as a smooth morphism.

There is much more to be said: first, this $\Omega_X(\log(X_s))/\Omega_S(\log(s))$ can be extended to a logarithmic de Rham complex, and its hypercohomology, while not recovering the cohomology of the singular fibers, does give rise to the limiting Hodge structure. So it is evidently worth considering.

Second, the picture is quite a bit more general, and can be applied to all toric and toroidal maps between toric varieties or toroidal embeddings (with a little caveat about the characteristic of the residue fields). So there is some flexibility in choosing $X \rightarrow S$.

1.3. The search for a structure

Since we are considering moduli, then as soon as we consider $X \rightarrow S$ as above we must also consider the normal crossings fiber $X_s \rightarrow \{s\}$. But what structure should we put on this variety? The notion of differentials with logarithmic poles along X_s is not in itself intrinsic to X_s . Also the normal crossings variety X_s is not in itself toric or toroidal, so a new structure is needed to incorporate it into the picture.

One is tempted to consider varieties which are assembled from nice variety by some sort of gluing, as normal crossings varieties are. But already normal crossings varieties do not give a satisfactory answer in general, because their deformation

spaces have “bad” components. Here is a classical example: consider a smooth projective variety Z such that $Pic^0(Z)$ is nontrivial. Let L be a line bundle on Z and set $Y = \mathbb{P}(\mathcal{O} \oplus L)$, with zero section $Z \subset Y$. Let X be the blowing up of $Z \times 0 \subset Y \times \mathbb{A}^1$. We have a flat morphism $f : X \rightarrow \mathbb{A}^1$ with fiber $X_0 = f^{-1}(0) \simeq Y \cup Y$, where the two copies of Y are glued with the zero section of one attached to the ∞ section of the other.

So clearly X_0 is a normal crossings variety with a nice smoothing to a copy of Y . But there are other deformations: the variety $Y \cup Y$ also deforms to $Y \cup Y'$ where $Y' = \mathbb{P}(\mathcal{O} \oplus L')$ and L' a deformation of the line bundle L . And it is not hard to see that $Y \cup Y'$ does not have a smoothing. Ideally one really does not want to see this deformation $Y \cup Y'$ in the picture - and ideally X_0 should have a natural structure whose deformation space excludes $Y \cup Y'$ automatically.

Such a structure was proposed by Friedman in [9], where the notion of *d-semistable varieties* was introduced. This structure is somewhat subtle, and while it solves the issue in this case, it is not quite as flexible as one could wish. As we will see in Section 5, logarithmic structures subsume d-semistability and do provide an appropriate flexibility.

1.4. Organization of this chapter

The purpose of this chapter is to briefly describe logarithmic structure and to indicate where they can be useful in the study of moduli spaces. Section 2 gives the basic definitions of logarithmic structures, and section 3 discusses logarithmic differentials and log smooth deformations, which are important in considering moduli spaces.

Section 4 gives the first example where logarithmic geometry fits well with moduli spaces: the moduli space of stable curves is the moduli space of log smooth curves. The issue of d-semistability does not arise since a nodal curve is automatically d-semistable. So the theory for curves is simple. Turning to higher dimensions, Section 5 shows how d-semistability can be described ~~within~~ logarithmic structures. unig.

If one is to enlarge algebraic geometry to include logarithmic structures, the task of generalizing the techniques of algebraic geometry to logarithmic structure can certainly seem daunting. In section 6 we show how to encode logarithmic structure in terms of certain algebraic stacks. This allows us to reduce various constructions to the case of algebraic stacks. (One can argue that the theory of stacks is not simple either, but at least in the theory of moduli they have come to be accepted, with some exceptions [30].)

In section 7 we make use of logarithmic stacks to describe the complexes which govern deformations and obstructions for logarithmic structures even in the non-smooth case. This comes in handy later. For instance, even when studying moduli

of log-smooth schemes, the moduli spaces tend to be singular, and their cotangent complexes are a necessary ingredients in constructing virtual fundamental classes.

Section 8 describes a beautiful construction, similar to polar coordinates, in which families of complex log smooth varieties give rise canonically to families of topological manifolds. Differential geometers have used polar coordinates on nodal curves to “make space” for monodromy to act by Dehn twists. Rounding (using Ogus’s terminology) is a magnificent way to generalize this.

The immediate implications of logarithmic structures for De Rham cohomology and Hodge structures are described in Section 9.

We conclude by describing three applications, where logarithmic structures serve as the proverbial “magic powder” (term suggested by Kato and Ogus) to clarify or remove unwanted behavior from moduli spaces.

Section 10 describes a number of cases where the main irreducible component of a moduli space can be separated from other “unwanted” components by sprinkling the objects with a bit of logarithmic structures.

In Section 11 we introduce twisted curves, a central object of orbifold stable maps, and show how logarithmic structures give a palatable way to construct the moduli stack of twisted curves.

Section 12 gives background for the work of B. Kim, in which Jun Li’s moduli space of relative stable maps, with its obstruction theory and virtual fundamental class, is beautifully simplified using logarithmic structures.

1.5. Notation

Following the lead of Ogus [40], we try whenever possible to denote a logarithmic scheme by a regular letter (such as X) and the underlying scheme by \underline{X} . When this is impossible we write X for the underlying scheme and (X, \mathcal{M}_X) for a logarithmic scheme over it.

1.6. Acknowledgements

This chapter originated from lectures given by Olsson at the School and Workshop on Aspects of Moduli, June 15-28, 2008 at the De Giorgi Center at the Scuola Normale Superiore in Pisa, Italy. The material was revisited and expanded in our seminar during the Algebraic Geometry program at MSRI, 2009. We thank the De Giorgi Center, MSRI, their staff and program organizers for providing these opportunities. Thanks are due to Arthur Ogus and Phillip Griffiths, who lectured on two topics at the MSRI seminar. While no new material is intended here, we acknowledge that research by Abramovich, Gillam and Olsson is supported by the NSF. Finally thanks are due to Gavril Farkas and Ian Morrison for the invitation to write this chapter.

2. Definitions and basic properties

Qile1

In this section we introduce the basic definitions of logarithmic geometry in the sense of [24]. Good introductions are given in [24] and [40]. Further technique is developed in [10].

2.1. Logarithmic structures

The basic definitions are as follows:

Definition 2.1. A *monoid* is a commutative semi-group with a unit. A morphism of monoids is required to preserve the unit element. We use Mon to denote the category of Monoids.

Def:log-str

Definition 2.2. Let \underline{X} be a scheme. A *pre-logarithmic structure* on \underline{X} is a sheaf of monoids \mathcal{M}_X on the étale site $\underline{X}_{\text{ét}}$ combined with a morphism of sheaves of monoids: $\alpha : \mathcal{M}_X \rightarrow \mathcal{O}_X$, called the structure morphism, where we view \mathcal{O}_X as a monoid under multiplication. A pre-log structure is called a *log structure* if $\alpha^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$ via α . The pair $(\underline{X}, \mathcal{M}_X)$ is called a *log scheme*, and will be denoted by X .

Remark that we view \mathcal{O}_X^* as subsheaf of \mathcal{M}_X .

Def:chara

Definition 2.3. Given a log scheme X , the quotient sheaf $\overline{\mathcal{M}}_X = \mathcal{M}_X / \mathcal{O}_X^*$ is called the *characteristic of the log structure* \mathcal{M}_X .

Definition 2.4. Let \mathcal{M} and \mathcal{N} be pre-log structures on \underline{X} . A *morphism* between them is a morphism $\mathcal{M} \rightarrow \mathcal{N}$ of sheaves of monoids which is compatible with the structure morphisms.

How should one think of such a beast? There are two extreme cases:

- (1) If an element $m \in \mathcal{M}$ has $\alpha(m) = x \neq 0$, one often thinks of m as some sort of partial data of a “branch of the logarithm of x ”. Evidently no data is added if x is invertible, but some is added otherwise. In particular, we will see later that m permits us to take the logarithmic differential dx/x of x .
- (2) If $\alpha(m) = 0$ it is often the case that it m comes by restricting the log structure of an ambient space, and serves as the “ghost” of a logarithmic cotangent vector coming from that space. So the log structure “remembers” deformations that are lost when looking at the underlying scheme.

2.2. The log structure associated to a pre-log structure

We have a natural inclusion

$$i : (\text{log structures on } \underline{X}) \hookrightarrow (\text{pre-log structures on } \underline{X})$$

by viewing a log structure as a pre-log structure. We now construct a left adjoint.

Let $\alpha : \mathcal{M} \rightarrow \mathcal{O}_X$ be a pre-log structure on X . We define the *associated log structure* \mathcal{M}^a to be the push-out of

$$\begin{array}{ccc} \alpha^{-1}(\mathcal{O}_X^*) & \longrightarrow & \mathcal{M} \\ \downarrow & & \\ \mathcal{O}_X^* & & \end{array}$$

in the category of sheaves of monoids on $X_{\text{ét}}$, endowed with

$$\mathcal{M}^a \rightarrow \mathcal{O}_X \quad (a, b) \mapsto \alpha(a)b \quad (a \in \mathcal{M}, b \in \mathcal{O}_X^*).$$

In this way, we obtain a functor $a : (\text{pre-log structures on } X) \rightarrow (\text{log structures on } X)$. From the universal property of push-out, any morphism of pre-log structure from a pre-log structure \mathcal{M} to a log structure on X factor through \mathcal{M}^a uniquely.

We have the following lemma:

Lemma 2.5. [40, 1.1.5] *The functor a is left adjoint to i .*

Example 2.6. The category of log structures on X has an initial object, called the trivial log structure, given by the inclusion $\mathcal{O}_X^* \hookrightarrow \mathcal{O}_X$. It also has a final object, given by the identity map $\mathcal{O}_X \rightarrow \mathcal{O}_X$. Trivial log structures are quite useful as they make the category of schemes into a full subcategory of the category of log schemes (see Definition 2.9). The final object is rarely used since it is not fine, see definition 2.16.

NClog

Example 2.7. Let X be a regular scheme, $D \subset X$ is a divisor. We can define a log structure \mathcal{M} on X associated to the divisor D as

$$\mathcal{M}(U) = \{g \in \mathcal{O}_X(U) : g|_{U \setminus D} \in \mathcal{O}_X^*(U \setminus D)\} \subset \mathcal{O}_X(U).$$

The case where D is a normal crossings divisor situation is special - we will see later that it is *log smooth*.

Note that the concept of normal crossing is local in the étale topology. This is one reason we use the étale topology instead of the Zariski topology.

Afflog

Example 2.8. Let P be a monoid, R a ring, and denote by $R[P]$ the monoid algebra. Denote $X = \text{Spec } R[P]$. then X has a canonical log structure associated to the canonical map $P \rightarrow R[P]$. We denote by $\text{Spec}(P \rightarrow R[P])$ the log scheme with underlying X , and the canonical log structure.

2.3. The inverse image and the category of log schemes

Let $f : X \rightarrow Y$ be a morphism of schemes. Given a log structure \mathcal{M}_Y on Y , we can define a log structure on X , called the *inverse image* of \mathcal{M}_Y , to be the log structure associated to the pre-log structure $f^{-1}(\mathcal{M}_Y) \rightarrow f^{-1}(\mathcal{O}_Y) \rightarrow \mathcal{O}_X$. This is usually denoted by $f^*(\mathcal{M}_Y)$. Using the inverse image of log structures, we can give the following definition.

) restatement.
wording.

Def: log-mor

Definition 2.9. A morphism of log schemes $X \rightarrow Y$ consists of a morphism of underlying schemes $f : \underline{X} \rightarrow \underline{Y}$, and a morphism $f^b : f^* \mathcal{M}_Y \rightarrow \mathcal{M}_X$ of log structures on \underline{X} .

We denote by $LSch$ the category of log schemes.

Example 2.10. In Example 2.8, the log structure on $\text{Spec}(P \rightarrow R[P])$ can be viewed as the inverse image of the log structure on $\text{Spec}(P \rightarrow \mathbb{Z}[P])$ via the canonical map $\text{Spec}(R[P]) \rightarrow \text{Spec}(\mathbb{Z}[P])$.

Logpt

Example 2.11. Let k be a field, $\underline{Y} = \text{Spec } k[x_1, \dots, x_n]$, $D = V(x_1 \cdots x_r)$. Note that D is a normal crossing divisor in \underline{Y} . By example 2.7, we have a log structure \mathcal{M}_Y on \underline{Y} associated to the divisor D . In fact, \mathcal{M}_Y can be viewed as a subsheaf of $\mathcal{O}_{\underline{Y}}$ generated by $\mathcal{O}_{\underline{Y}}^*$ and $\{x_1, \dots, x_r\}$.

Consider the inclusion $j : p = \text{Spec } k \hookrightarrow \underline{Y}$ sending the point to the origin of \underline{Y} . Then $j^* \mathcal{M}_Y = k^* \oplus \mathbb{N}^r$, and the structure map $j^* \mathcal{M} \rightarrow \mathcal{O}_{\underline{X}}$ is given by $(a, n_1, \dots, n_r) \mapsto a \cdot 0^{n_1 + \dots + n_r}$, where we assume $0^0 = 1$ and $0^n = 0$ if $n \neq 0$. Such point with the log structure above is called a logarithmic point, when $r = 1$ we call it the standard logarithmic point.

define

2.4. Charts of log structures

Def: chart

Definition 2.12. Let X be a log scheme, and P a monoid. A chart for \mathcal{M}_X is a morphism $P \rightarrow \Gamma(X, \mathcal{M}_X)$, such that the log structure associated to the pre-log structure coming from the induced map $P \rightarrow \Gamma(X, \mathcal{M}_X) \rightarrow \Gamma(X, \mathcal{O}_X)$ is \mathcal{M}_X .

wording
 $P \rightarrow \mathcal{M}_X$

In fact, a chart of \mathcal{M}_X is equivalent to a morphism

$$f : X \rightarrow \text{Spec}(P \rightarrow \mathbb{Z}[P]),$$

such that f^b is an isomorphism. In general, we have the following:

Lemma 2.13. [40, 1.1.9] *The morphism*

$$\text{Hom}_{LSch}(X, \text{Spec}(P \rightarrow \mathbb{Z}[P])) \rightarrow \text{Hom}_{Mon}(P, \Gamma(X, \mathcal{M}_X))$$

associating to f the composition

$$P \longrightarrow \Gamma(P_X) \xrightarrow{\Gamma(f^b)} \Gamma(\mathcal{M}_X)$$

is an isomorphism.

We can also consider charts for log morphisms.

Def: chart-mor

Definition 2.14. Let $f : X \rightarrow Y$ be a morphism of log schemes. A chart for f is a triple $(P_X \rightarrow \mathcal{M}_X, Q_Y \rightarrow \mathcal{M}_Y, Q \rightarrow P)$ where P_X and Q_Y are the constant sheaves associated to the monoids P and Q , which satisfy the following conditions:

- (1) $P_X \rightarrow \mathcal{M}_X$ and $Q_Y \rightarrow \mathcal{M}_Y$ are charts of \mathcal{M}_X and \mathcal{M}_Y ;

(2) the morphism of monoids $Q \rightarrow P$ makes the following diagram commutative:

$$\begin{array}{ccc} Q_X & \longrightarrow & P_X \\ \downarrow & & \downarrow \\ f^* \mathcal{M}_Y & \longrightarrow & \mathcal{M}_X. \end{array}$$

2.5. Fine log structures

Arbitrary log structures are too wild to manipulate, in rough analogy to arbitrary ringed spaces: they are useful for general constructions but a narrower, more geometric category is desirable. In Definition 2.16 below we introduce some well-behaved log structures, which are more in analogy to noetherian schemes: you can do some geometry on them.

wording,

Given a monoid P , we can associate a group

$$P^{gp} := \{(a, b) \mid (a, b) \sim (c, d) \text{ if } \exists s \in P \text{ such that } s + a + d = s + b + c\}.$$

Note that any morphism from P to an abelian group factors through P^{gp} uniquely.

Def: integral

Definition 2.15. P is called *integral* if $P \rightarrow P^{gp}$ is injective. It is called *saturated* if it is integral and for any $p \in P^{gp}$, if $n \cdot p \in P$ for some positive integer n then $p \in P$.

Def: fine

Definition 2.16. A log scheme X is said to be *fine*, if étale locally there is a chart $P \rightarrow \mathcal{M}_X$ with P a finitely generated integral monoid. If moreover P is saturated, X is called a *fine and saturated* (or *fs*) log structure. This is equivalent to saying that for every geometric point $\bar{x} \rightarrow \underline{X}$ the monoid $\overline{\mathcal{M}}_{\bar{x}, X}$ is saturated. Finally if $P \simeq \mathbb{N}^k$ we say that the log structure is *locally free*.

can be chosen

ref.

In the following, we will focus on fine log schemes.

3. Differentials, smoothness, and log smooth deformations

Qile2

3.1. Logarithmic differentials

In [12] Grothendieck defines a derivation as the difference of infinitesimal liftings of a section. We can do the same thing with logarithmic schemes. First, we need a concept of infinitesimal extension, which requires the following definition.

Definition 3.1. A morphism $f : X \rightarrow Y$ of log schemes is called *strict* if $f^\flat : f^* \mathcal{M}_Y \rightarrow \mathcal{M}_X$ is an isomorphism. It is called a *strict closed immersion*¹ if it is strict and the underlying map $\underline{X} \rightarrow \underline{Y}$ is a closed immersion in the usual sense.

¹The term used in [24] is *an exact closed immersion*.

Let us consider a commutative diagram of solid arrows of log schemes:

$$\begin{array}{ccc}
 T_0 & \xrightarrow{\phi} & X \\
 \downarrow j & \nearrow g_1 & \downarrow f \\
 T_1 & \xrightarrow{\psi} & Y
 \end{array}$$

where j is a strict closed immersion defined by an ideal J with $J^2 = 0$. Note that T_0 and T_1 have the same underlying topological space. Then we have the following commutative diagram of sheaves of algebras given by the underlying maps:

and isomorphic sheaves

$$\begin{array}{ccc}
 \mathcal{O}_{T_0} & \xleftarrow{\phi^{-1}} & \phi^{-1}\mathcal{O}_X \\
 \uparrow & \nearrow g_1^\# & \uparrow \\
 \mathcal{O}_{T_1} & \xleftarrow{\psi^{-1}} & \psi^{-1}\mathcal{O}_Y
 \end{array}$$

Then $g_1^\# - g_2^\#$ is a derivation $\partial_{g_1-g_2} : \phi^{-1}\mathcal{O}_X \rightarrow J$ in the usual sense. We also have a commutative diagram given by the log structures:

$$\begin{array}{ccc}
 \mathcal{M}_{T_0} & \xleftarrow{\phi^{-1}} & \phi^{-1}\mathcal{M}_X \\
 \uparrow & \nearrow g_1^b & \uparrow \\
 \mathcal{M}_{T_1} & \xleftarrow{\psi^{-1}} & \psi^{-1}\mathcal{M}_Y
 \end{array}$$

Note that we have an "exact sequence" of multiplicative monoids

$$1 \rightarrow (1 + J) \rightarrow \mathcal{M}_{T_1} \rightarrow \mathcal{M}_{T_0} \rightarrow 1,$$

by which we mean that the group $1 + J$ acts freely on \mathcal{M}_{T_1} with quotient \mathcal{M}_{T_0} . Hence we obtain a morphism $D_{g_1-g_2} : \phi^{-1}\mathcal{M}_X \rightarrow J$ such that for every $m \in \phi^{-1}\mathcal{M}_X$ we have $(g_1^b - g_2^b)(m) = 1 + D_{g_1-g_2}(m)$. It is not hard to check that it is a group homomorphism: $D_{g_1-g_2}(m \cdot n) = D_{g_1-g_2}(m) + D_{g_1-g_2}(n)$ for any $m, n \in \phi^{-1}(\mathcal{M}_X)$. By the definition of log structures, we also have

- (1) $\alpha(m)D_{g_1-g_2}m = \partial_{g_1-g_2}(\alpha(m)), \forall m \in \phi^{-1}\mathcal{M}_X;$
- (2) $D_{g_1-g_2}|_{\psi^{-1}\mathcal{M}_Y} = 0.$

Remark 3.2 (1) Since the log structure contains all the invertible elements in the structure sheaf, the map $D_{g_1-g_2}$ determines $\partial_{g_1-g_2}$.

- (2) The above properties show that $D_{g_1-g_2}$ behaves like "d log". This is one of the reasons for the name "logarithmic structure".

Summarizing the above discussion gives the following definitions:

LogDer

Definition 3.3. Consider $f : X \rightarrow Y$ a morphism of fine log schemes. Let I be an \mathcal{O}_X -module. A log derivation of X over Y to I is a pair (∂, D) where $\partial \in \text{Dery}_X(X, I)$ and $D : \mathcal{M}_X \rightarrow I$ such that the following conditions hold:

is additive map
insert ref to Ogus, Furushita, Griffiths trans...

- (1) $D(ab) = D(a) + D(b)$ for $a, b \in \mathcal{M}_X$;
- (2) $\alpha(a)D(a) = \partial(\alpha(a))$, for $a \in \mathcal{M}_X$.
- (3) $D(a) = 0$, for $a \in f^{-1}\mathcal{M}_Y$.

The sheaf $Der_Y(X, I)$ of log derivations of X over Y to I is the sheaf of germs of pairs (∂, D) . The sheaf $Der_Y(X, \mathcal{O}_X)$ is usually denoted by $T_{X/Y}$, and is called the logarithmic tangent sheaf of X over Y .

As an analogue of differentials of usual schemes, we have the following result:

Proposition 3.4. [40, IV.1.1.6] *There exists an \mathcal{O}_X -module $\Omega_{X/Y}^1$ with a universal derivations $(\partial, D) \in Der_Y(X, \Omega_{X/Y}^1)$, such that for any \mathcal{O}_X -module I , the canonical map*

$$Hom_{\mathcal{O}_X}(\Omega_{X/Y}^1, I) \rightarrow Der_Y(X, I), \quad u \mapsto (u \circ \partial, u \circ D)$$

is an isomorphism of \mathcal{O}_X -modules. In fact, we have the following construction:

$$\Omega_{X/Y}^1 = \Omega_{X/Y} \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{gp}) / \mathcal{K}$$

where \mathcal{K} is a \mathcal{O}_X -module generated by local sections of the following forms:

- (1) $(d\alpha(a), 0) - (0, \alpha(a) \otimes a)$ with $a \in \mathcal{M}_X$;
- (2) $(0, 1 \otimes a)$ with $a \in Im(f^{-1}(\mathcal{M}_Y) \rightarrow \mathcal{M}_X)$.

The universal derivation (∂, D) is given by $\partial : \mathcal{O}_X \xrightarrow{d} \Omega_{X/Y} \rightarrow \Omega_{X/Y}^1$ and $D : \mathcal{M}_X \rightarrow \mathcal{O}_X \otimes_{\mathbb{Z}} \mathcal{M}_X^{gp} \rightarrow \Omega_{X/Y}^1$.

LodDiff

Definition 3.5. Given a morphism $f : X \rightarrow Y$ of log schemes, the \mathcal{O}_X -module $\Omega_{X/Y}^1$ is called the sheaf of logarithmic differentials. Sometimes we use the short notation Ω_f^1 for $\Omega_{X/Y}^1$.

Note that the dual sheaf of $\Omega_{X/Y}^1$ is $T_{X/Y}$.

$$Hom(\Omega_{X/Y}^1, \mathcal{O}_X) \cong T_{X/Y}$$

Remark 3.6. If we consider only fine log structures, and assume that Y is locally noetherian and X locally of finite type over Y , then both $Der_Y(X, I)$ and $\Omega_{X/Y}^1$ in the definitions above are coherent sheaves. The proof of this can be found in [40, IV.1.1]

NCsmooth

Example 3.7. Let $R = k[x_1, \dots, x_n]/(x_1 \cdots x_r)$, where k is a field. Denote $X = Spec R$, and \mathcal{M}_X to be the log structure on X given by $\mathbb{N}^r \rightarrow R$, $e_i \mapsto x_i$, where e_i is the standard generator of the monoid \mathbb{N}^r . Let $Y = Spec(\mathbb{N} \rightarrow k)$ be the logarithmic point described in 2.11. Now we can define a morphism $f : X \rightarrow Y$ by the following diagram:

wording

$$\begin{array}{ccc} \mathbb{N}^r & \longrightarrow & R \\ \uparrow \Delta & & \uparrow \\ \mathbb{N} & \longrightarrow & k \end{array}$$

where $\Delta : e \mapsto e_1 + \dots + e_r$, and e is the standard generator of \mathbb{N} . Then it is easy to see that $Der_Y(X, \mathcal{O}_X)$ is a free \mathcal{O}_X -module generated by

$$x_1 \frac{\partial}{\partial x_1}, \dots, x_r \frac{\partial}{\partial x_r}, \frac{\partial}{\partial x_{r+1}}, \dots, \frac{\partial}{\partial x_n},$$

with a relation $x_1 \frac{\partial}{\partial x_1} + \dots + x_r \frac{\partial}{\partial x_r} = 0$. The sheaf Ω_f^1 is a free \mathcal{O}_X -module generated by the logarithmic differentials:

$$\frac{dx_1}{x_1}, \dots, \frac{dx_r}{x_r}, dx_{r+1}, \dots, dx_n$$

with a relation $\frac{dx_1}{x_1} + \dots + \frac{dx_r}{x_r} = 0$.

Example 3.8. Let $h : Q \rightarrow P$ be a morphism of fine monoids. Denote $X = \text{Spec}(P \rightarrow \mathbb{Z}[P])$ and $Y = \text{Spec}(Q \rightarrow \mathbb{Z}[Q])$. Then we have a morphism $f : X \rightarrow Y$ induced by h . A direct calculation shows that $\Omega_f^1 = \mathcal{O}_X \otimes \text{Cok}(h^{gp})$.

(This can also be seen from universal property).

3.2. Logarithmic Smoothness

Let us go back to the following cartesian diagram of log schemes:

diag:smooth

(3.9)

$$\begin{array}{ccc} T_0 & \xrightarrow{\phi} & X \\ J \downarrow j & & \downarrow f \\ T_1 & \xrightarrow{\psi} & Y \end{array}$$

where j is a strict closed immersion defined by J with $J^2 = 0$. As in the usual case, we can define log smoothness by the infinitesimal lifting property.

defn:logsmooth

Definition 3.10. A morphism $f : X \rightarrow Y$ of fine log schemes is called *log smooth* (resp. *étale*) if the underlying morphism $\underline{X} \rightarrow \underline{Y}$ is locally of finite presentation and for any commutative diagram as (3.9), étale locally on T_1 there exists (resp. there exists a unique) morphism $g : T_1 \rightarrow X$ such that $\phi = g \circ j$ and $\psi = f \circ g$.

wording

We have the following useful criterion for smoothness from [24, Theorem 3.5].

KatoStrThm

Theorem 3.11. (*K.Kato*) Let $f : X \rightarrow Y$ be a morphism of fine log schemes. Assume we have a chart $Q \rightarrow \mathcal{M}_Y$, where Q is a finitely generated integral monoid. Then the following are equivalent:

- (1) f is log smooth (resp. log étale);
- (2) étale locally on X , there exists a chart $(P_X \rightarrow \mathcal{M}_X, Q_Y \rightarrow \mathcal{M}_Y, Q \rightarrow P)$ extending the chart $Q_Y \rightarrow \mathcal{M}_Y$, satisfying the following properties.
 - (a) The kernel and the torsion part of the cokernel (resp. the kernel and the cokernel) of $Q^{gp} \rightarrow P^{gp}$ are finite groups of order invertible on X .

(b) *The induced morphism from $\underline{X} \rightarrow \underline{Y} \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P]$ is étale in the classical sense.*

Remark 3.12. (1) We can require $Q^{gp} \rightarrow P^{gp}$ in (a) to be injective, and replace the requirement of $\underline{X} \rightarrow \underline{Y} \times_{\text{Spec } \mathbb{Z}[Q]} \text{Spec } \mathbb{Z}[P]$ be étale in (b) by requiring it to be smooth without changing the conclusion of the theorem 3.11.

(2) The arrow in (b) shows that a log smooth morphism is “locally toric” relative to the base. Consider the case Y is a log scheme with underlying space given by $\text{Spec } \mathbb{C}$ with the trivial log structure, and $X = \text{Spec}(P \rightarrow \mathbb{C}[P])$ where P is a fine, saturated and torsion free monoid. Then \underline{X} is a toric variety with the action of $\text{Spec } \mathbb{C}[P^{gp}]$. According to the theorem, X is log smooth relative to Y , though the underlying space might be singular. These singularities are called toric singularities in [25]. This is closely related to the classical notion of toroidal embeddings, see [28].

Example 3.13. Using the theorem, we can check directly that the morphism f in example 3.7 is log smooth, but the underlying map has normal crossing singularities. We will see later that one of the major advantages of log structures is in dealing with the normal crossing singularities.

Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of fine log schemes. Consider the sheaves of log differentials Ω_g^1 and $\Omega_{g \circ f}^1$, with their universal derivations (∂_g, D_g) and $(\partial_{g \circ f}, D_{g \circ f})$ respectively. We have a canonical map $f^* \Omega_g^1 \rightarrow \Omega_{g \circ f}^1$ induced by

$$f^*(\partial_g u) \mapsto \partial_{g \circ f} f^*(u) \text{ and } f^*(D_g v) \mapsto D_{g \circ f} f^*(v),$$

where $u \in \mathcal{O}_Y$ and $v \in \mathcal{M}_Y$. Denote by (∂_f, D_f) the universal derivation associated to Ω_f^1 . Similarly, we have a canonical map $\Omega_{g \circ f}^1 \rightarrow \Omega_f^1$ induced by

$$\partial_{g \circ f} u' \mapsto \partial_f u' \text{ and } D_{g \circ f} v' \mapsto D_f v',$$

where $u' \in \mathcal{O}_X$ and $v' \in \mathcal{M}_X$. The following proposition shows that log differentials behave like usual differentials, especially for log smooth morphisms.

p:logsmoothdif

Proposition 3.14. (1) *The sequence $f^* \Omega_g^1 \rightarrow \Omega_{g \circ f}^1 \rightarrow \Omega_f^1 \rightarrow 0$ is exact.*

(2) *If f is log smooth, then Ω_f^1 is a locally free \mathcal{O}_X -module, and we have the following exact sequence: $0 \rightarrow f^* \Omega_g^1 \rightarrow \Omega_{g \circ f}^1 \rightarrow \Omega_f^1 \rightarrow 0$.*

(3) *If $g \circ f$ is log smooth and the sequence in (2) is exact and splits locally, then f is log smooth.*

A proof can be found in [40, Chapter IV].

3.3. Logarithmic smooth deformation

Having discussed log smoothness, a natural thing to do is to develop log smooth deformations. In many cases, we would require this to be a flat deformation for the underlying space. Unfortunately log smoothness does not imply flatness, so we need the following definition.

integralmonmap

Definition 3.15. A map of fine monoids $h : Q \rightarrow P$ is called *integral* if the induced map on monoid algebra $\mathbb{Z}[Q] \rightarrow \mathbb{Z}[P]$ is flat.

fn:integralmap

Definition 3.16. A morphism $f : X \rightarrow Y$ of integral log schemes is called *integral* if for every geometric point $\bar{x} \in X$, the map of characteristic monoids $h : f^{-1}(\overline{\mathcal{M}}_Y)_{\bar{x}} \rightarrow \overline{\mathcal{M}}_{X,\bar{x}}$ is integral.

line?

Remark 3.17. (1) If f is integral, then étale locally we have a chart $(P_X \rightarrow \mathcal{M}_X, Q_Y \rightarrow \mathcal{M}_Y, Q \xrightarrow{h} P)$ such that h is integral.

(2) If $h : Q \rightarrow P$ is integral of integral monoids, then for any integral monoid Q' , the push-out of $P \leftarrow Q \rightarrow Q'$ in the category of monoids is integral. Thus integral morphisms are stable under base change by integral log schemes.

(3) Given a morphism $h : Q \rightarrow P$ of integral monoids, there is an explicit criterion, which looks complicated, but sometimes is useful for checking integrality of h directly: if $a_1, a_2 \in Q$, $b_1, b_2 \in P$ and $h(a_1)b_1 = h(a_2)b_2$, then there exist $a_3, a_4 \in Q$ and $b \in P$ such that $b_1 = h(a_3)b$, $b_2 = h(a_4)b$ and $a_1a_3 = a_2a_4$. This comes essentially from the equational criterion for flatness.

Now we have the following fact from [24, 4.5].

logsmooth-flat

Proposition 3.18. If f is a log smooth and integral morphism of fine log schemes, then f the underlying map is flat in the usual sense.

Now let us consider the following deformation problem. We are given a log smooth integral morphism $f_0 : X_0 \rightarrow B_0$ of fine log schemes, and a strict closed immersion $j : B_0 \rightarrow B$ defined by an ideal J with $J^2 = 0$. We want to find a log smooth lifting $f : X \rightarrow B$ fitting in the following cartesian diagram:

$$\begin{array}{ccc} X_0 & \hookrightarrow & X \\ \downarrow & & \downarrow \\ B_0 & \hookrightarrow & B \end{array}$$

Remark 3.19. Since f_0 is integral, and $\mathcal{M}_X/(1+J) \cong \mathcal{M}_{X_0}$, it is not hard to show that the lifting f is automatically integral and hence flat.

We have the following theorem for log smooth deformations.

hm:basiclogdef

Theorem 3.20. With the notation as above, we have:

- (1) There is a canonical obstruction $\eta \in H^2(X_0, T_{X_0/B_0} \otimes J)$ such that $\eta = 0$ if and only if there exists a log smooth lifting.
- (2) If $\eta = 0$, then the set of log smooth deformations form a torsor under $H^1(X_0, T_{X_0/B_0} \otimes J)$.

ref to Kato

- (3) *The automorphism group of any deformation is isomorphic to $H^0(X_0, T_{X_0/B_0} \otimes J)$.*

The theorem can be proved in a manner similar to the case of usual deformation theory as in [13, Exposé 3]. Another proof using the logarithmic cotangent complex can be found in [45, Thm 5.6], which we will discuss later.

4. Log smooth curves and their moduli

Satriano

In this section we discuss F. Kato's paper [23] in which he gives a log geometry theoretic construction of the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ of the moduli space $\mathcal{M}_{g,n}$ of curves. He states in the introduction a motivating philosophy which relates log geometry to compactifications of moduli spaces:

Philosophy. Since log smoothness already incorporates degenerate objects, one should expect that a moduli space of log smooth objects is already compact, and so provides a compactification of the moduli of objects where the log structure is trivial.

Along the lines of this philosophy, to compactify $\mathcal{M}_{g,n}$, we want to introduce a notion of *log curve* which extends the notion of smooth curve. Following F. Kato, we do so after some preliminaries.

4.1. Relative characteristic sheaves

Recall from Definition 2.3 that the characteristic $\overline{\mathcal{M}}_X$ of a log scheme X is defined as $\mathcal{M}_X/\mathcal{O}_X^*$. In the study of log curves, the following relative notion of characteristic plays an important role.

def:char

Definition 4.1. Given a morphism $f : X \rightarrow Y$ of log schemes, the *relative characteristic* $\overline{\mathcal{M}}_{X/Y}$ is defined as the quotient $\mathcal{M}_X/\text{im}(f^*\mathcal{M}_Y \rightarrow \mathcal{M}_X)$ in the category of integral monoids.

ex:char

Example 4.2. Let $f : X \rightarrow Y$ be the morphism from Example 3.7. Then the relative characteristic $\overline{\mathcal{M}}_{X/Y}$ is the cokernel in the category of integral monoids of the diagonal map $\Delta : \mathbb{N} \rightarrow \mathbb{N}^2$, which is \mathbb{Z} .

Lemma 4.3 ([23, Lemma 1.6]). *If $f : X \rightarrow Y$ is an integral morphism of fine log schemes, then $\overline{\mathcal{M}}_{X/Y, \bar{x}} = 0$ if and only if f is strict in an étale neighborhood of x .*

As the following example illustrates, the integrality assumption on f is necessary.

ex:intrel

Example 4.4. Let P be the monoid on three generators x, y , and z subject to the relation $x + y = 2z$. We have an injection

$$i : P \rightarrow \mathbb{N}^2$$

Better lie in

sending x to $(2, 0)$, y to $(1, 1)$, and z to $(0, 2)$. Let $X = \text{Spec } k[\mathbb{N}^2]$ and $Y = \text{Spec } k[P]$ with their canonical log structures. Then it follows from [24, Prop 3.4] that the morphism of log schemes $f : X \rightarrow Y$ induced by i is log étale, but f is not flat, and hence, by [24, Cor 4.5], f is not integral. It is easy to check that

$$\mathcal{M}_{X/Y,0} = \mathbb{N}^2/P \simeq \mathbb{Z}/2,$$

so $\overline{\mathcal{M}}_{X/Y,0} = 0$, but f is not strict.

4.2. Log curves

bsec:logcurves
def:logcurve

Definition 4.5. A log curve is a log smooth integral morphism $f : X \rightarrow S$ of fs log schemes such that the geometric fibers of f are reduced connected curves. def?

We require f to be integral so that, by [24, Cor 4.5], f is flat. The reason for the fs assumption is to avoid cusps or worse singularities, as the following example shows.

ex: cusp

Example 4.6. If $X = \text{Spec } k[\mathbb{N} - \{1\}]$ is given its canonical log structure and $S = \text{Spec } k$ is given the trivial log structure, then $X \rightarrow S$ is log smooth and integral; however, $\underline{X} = \text{Spec } k[x, y]/(y^2 - x^3)$ has a cusp.

It is a remarkable fact that by endowing our curves with log structures as in Definition 4.5, this is enough to control the singularities of the curve.

thm: node

Theorem 4.7 ([23, Thm 1.3]). *If k is a separably closed field and $f : X \rightarrow S$ is a log curve with $\underline{S} = \text{Spec } k$, then \underline{X} has at worst nodal singularities. Moreover, if r_1, \dots, r_ℓ are the nodes of \underline{X} , then there exist smooth points s_1, \dots, s_n of \underline{X} such that*

$$\overline{\mathcal{M}}_{X/S} = \mathbb{Z}_{r_1} \oplus \dots \oplus \mathbb{Z}_{r_\ell} \oplus \mathbb{N}_{s_1} \oplus \dots \oplus \mathbb{N}_{s_n};$$

here \mathcal{M}_x denotes the skyscraper sheaf for a monoid M supported at a point $x \in \underline{X}$.

The reader should think of the s_i in the above theorem as marked points. So we can already see how n -pointed curves emerge naturally from the log geometry perspective.

ex: logcurve

Example 4.8. Consider the closed subscheme \underline{X} of $\mathbb{P}_k^2 \times_k \mathbb{A}_k^1$ defined by $xz = ty$, where t is the coordinate of \mathbb{A}_k^1 and x, y, z are the coordinates of \mathbb{P}_k^2 . Then \underline{X} has a natural log structure \mathcal{M}_X . For example, on the locus where z is invertible, \underline{X} is given by $\text{Spec } k[P_z]$ with P_z a monoid on five generators a, b, c, c', u subject to the relations $c + c' = 0$ and $a + c = b + u$; here \mathcal{M}_X is given by the canonical log structure associated to P_z . Then the projection

$$X \longrightarrow \mathbb{A}_k^1$$

is a log curve, where X is given the log structure above and \mathbb{A}_k^1 is given the log structure defined by the divisor $t = 0$. We see that every fiber above $t \neq 0$ is isomorphic to \mathbb{P}_k^1 with log structure given by the divisor at 0 and ∞ ; the fiber above $t = 0$ is nodal. The n in Theorem 4.7 is equal to 2 for all geometric fibers.

Since our goal is to give a log geometric description of $\overline{\mathcal{M}}_{g,n}$, we would like to express the stability condition purely in terms of log geometry. The following proposition provides the key.

prop:dualizing

Proposition 4.9 ([23, Prop 1.13]). *With notation as in Theorem 4.7, there is a natural isomorphism*

$$\Omega_{X/S}^1 \longrightarrow \omega_X(s_1 + \cdots + s_n),$$

where ω_X is the dualizing sheaf of X .

We therefore make the following definition.

def:stable

Definition 4.10. Let $f : X \rightarrow S$ be a log curve and for all geometric points \bar{t} of \underline{S} , let $\ell(\bar{t})$ and $n(\bar{t})$ be such that

$$\overline{\mathcal{M}}_{X_{\bar{t}}/\bar{t}} = \mathbb{Z}_{r_1} \oplus \cdots \oplus \mathbb{Z}_{r_{\ell(\bar{t})}} \oplus \mathbb{N}_{s_1} \oplus \cdots \oplus \mathbb{N}_{s_{n(\bar{t})}}.$$

We say f is of type (g, n) if f is proper, X has genus g , and $n(\bar{t}) = n$ for all \bar{t} . We say f is stable of type (g, n) if it is of type (g, n) and

$$H^0(X_{\bar{t}}, T_{X_{\bar{t}}/\bar{t}}) = 0$$

for all geometric points \bar{t} of \underline{S} .

It is, in fact, true ([23, Prop 1.7]) that if $f : X \rightarrow S$ is a log curve of type (g, n) , then the s_i in each geometric fiber fit together to yield n sections σ_i of f . It follows then that every stable log curve of type (g, n) is an n -pointed stable curve of genus g in the classical sense.

4.3. Log structures on stable curves

Having now shown that every log curve is naturally a pointed nodal curve, we shift gears and ask the following question: given a stable genus g curve $f : X \rightarrow S$ with n marked points, how many log structures can we put on X and S so that the associated morphism of log schemes is a log curve with relative characteristic supported on our given n marked points? We begin by constructing a canonical such log structure.

Lemmas 2.1 and 2.2 of [23] show that to endow X and S with log structures as desired, it is enough to consider the case when $\underline{S} = \text{Spec } A$ and A is strict Henselian. For every node r_i of the closed fiber of f , we can find an étale neighborhood U_i of the points specializing to r_i and a diagram

$$\begin{array}{ccc} U_i & \xrightarrow{\psi_i} & \text{Spec } A[x, y, t]/(xy - t) \\ \downarrow & & \downarrow \pi \\ S & \xrightarrow{\varphi_i} & \text{Spec } A[t] \end{array}$$

which is cartesian. Let $t_i \in A$ be the image of t under the morphism induced by φ_i . Endowing $\text{Spec } A[t]$ with the log structure associated to the morphism $\mathbb{N} \rightarrow A[t]$

sketching

sending 1 to t , and $\text{Spec } A[x, y, t]/(xy - t)$ with the log structure associated to $\mathbb{N}^2 \rightarrow A[x, y, t]/(xy - t)$ sending e_1 (resp. e_2) to x (resp. y), we see that (π, Δ) is a morphism of log schemes, where $\Delta : \mathbb{N} \rightarrow \mathbb{N}^2$ is the diagonal map. Pulling back these log structures under φ_i (resp. ψ_i), we obtain log structures \mathcal{L}_i (resp. \mathcal{M}'_i) on S (resp. U_i). Away from the points specializing to τ_i , we define a log structure \mathcal{M}''_i as the pullback of \mathcal{L}_i . The log structures \mathcal{M}'_i and \mathcal{M}''_i glue to yield a log structure \mathcal{M}_i on X . Let \mathcal{N} be the log structure on X associated to the divisor defined by the marked points. We let

$$\mathcal{M}_X = \mathcal{M}_1 \oplus_{\mathcal{O}_X^*} \cdots \oplus_{\mathcal{O}_X^*} \mathcal{M}_\ell \oplus_{\mathcal{O}_X^*} \mathcal{N}$$

and

$$\mathcal{M}_S = \mathcal{L}_1 \oplus_{\mathcal{O}_S^*} \cdots \oplus_{\mathcal{O}_S^*} \mathcal{L}_\ell.$$

It is not difficult to see that with these definitions, we have endowed f with the structure of a log curve.

Moreover, a detailed analysis of the proof of Theorem 4.7 shows that this log structure we have just constructed is “minimal” among all possible log structures giving $\underline{X}/\underline{S}$ the structure of a log curve (see 1.8 and Thm 2.3 of [23]):

thm:basic

Theorem 4.11. *Let $\underline{X}/\underline{S}$ be a stable genus g curve with n marked points and let X/S be the log curve obtained by endowing $\underline{X}/\underline{S}$ with the canonical log structure above. If X'/S' is a log curve and $\underline{a} : \underline{S}' \rightarrow \underline{S}$ and $\underline{b} : \underline{X}' \rightarrow \underline{X}$ are morphisms such that $\underline{X}' \simeq \underline{X} \times_{\underline{S}} \underline{S}'$ and such that the divisors of marked points in \underline{X}' are sent scheme-theoretically to the divisors of marked points in \underline{X} , there are unique morphisms a and b of log schemes extending the morphisms \underline{a} and \underline{b} above such that*

$$\begin{array}{ccc} X' & \xrightarrow{b} & X \\ \downarrow & & \downarrow \\ S' & \xrightarrow{a} & S \end{array}$$

is cartesian in the category of fs log schemes.

def:basic

Definition 4.12. A log curve X/S is called *basic* if it satisfies the universal property in Theorem 4.11.

4.4. Moduli

Before discussing moduli of log smooth curves, we begin with some generalities about log structures and stacks. Note that the definition of a log structure is not particular to schemes; indeed, Definition 2.2 makes sense for any ringed topos. We can therefore define log structures on the étale site of a Deligne-Mumford stack or the lisse-étale site of an Artin stack. The notions of fine and fs log structures carry over to this setting as well, so one can speak of fine (or fs) log algebraic stacks.

There is another equivalent way to talk about log structures on stacks; namely, if \mathfrak{X} is a stack over the category of schemes, and $\mathcal{M}_{\mathfrak{X}}$ is a log structure on \mathfrak{X} , then \mathfrak{X} can naturally be viewed as a stack over the category of log schemes. For concreteness, say \mathfrak{X} is a stack over the category of schemes with the étale topology. Then we obtain a category $\tilde{\mathfrak{X}}$ fibered over the category of log schemes by defining $\tilde{\mathfrak{X}}(S, \mathcal{M}_S)$ to be the category whose objects are morphisms

$$f : (S, \mathcal{M}_S) \rightarrow (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}})$$

of log stacks and whose morphisms g from $f \in \tilde{\mathfrak{X}}(S, \mathcal{M}_S)$ to $f' \in \tilde{\mathfrak{X}}(S', \mathcal{M}_{S'})$ are given by diagrams

$$\begin{array}{ccc} (S, \mathcal{M}_S) & \xrightarrow{g} & (S', \mathcal{M}_{S'}) \\ & \searrow f & \swarrow f' \\ & (\mathfrak{X}, \mathcal{M}_{\mathfrak{X}}) & \end{array}$$

where g is strict. One checks that $\tilde{\mathfrak{X}}$ is a stack over the category of log schemes with the strict étale topology.

Conversely, given any stack \mathcal{Y} over the category of log schemes with the strict étale topology, we obtain a log stack $(\mathcal{Y}', \mathcal{M}_{\mathcal{Y}'})$ over the category of schemes with the étale topology by letting $\mathcal{Y}'(S)$ be the category of pairs (\mathcal{M}_S, ξ) with ξ is an object of $\mathcal{Y}(S, \mathcal{M}_S)$. The log structure $\mathcal{M}_{\mathcal{Y}'}$ is then defined by the following property: if $f : S \rightarrow \mathcal{Y}$ is a morphism which corresponds to the pair (\mathcal{M}_S, ξ) , then $f^* \mathcal{M}_{\mathcal{Y}'} = \mathcal{M}_S$.

def?
Is this really what we want?

We now return to moduli of log curves. Let $\overline{\mathcal{M}}_{g,n}^{log}$ be the stack over the category of fs log schemes with the strict étale topology where $\overline{\mathcal{M}}_{g,n}^{log}(S, \mathcal{M}_S)$ is the category of stable log curves of type (g, n) over (S, \mathcal{M}_S) . By the above discussion, we obtain a log stack over the category of schemes with the étale topology. We again denote this log stack by $(\overline{\mathcal{M}}_{g,n}^{log}, \mathcal{M}_{\overline{\mathcal{M}}_{g,n}^{log}})$.

Note that the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$ carries a natural log structure $\mathcal{M}_{\overline{\mathcal{M}}_{g,n}}$ coming from the simple normal crossing divisor at the boundary. It follows from [7, §2] that $\mathcal{M}_{\overline{\mathcal{M}}_{g,n}}$ can also be described as the log structure which assigns to each stable curve $\underline{X}/\underline{S}$ the basic log structure obtained on \underline{S} . The results of Section 4.2 above then show that we have a natural morphism

$$F : (\overline{\mathcal{M}}_{g,n}^{log}, \mathcal{M}_{\overline{\mathcal{M}}_{g,n}^{log}}) \rightarrow (\overline{\mathcal{M}}_{g,n}, \mathcal{M}_{\overline{\mathcal{M}}_{g,n}})$$

and from Theorem 4.11 we see

Theorem 4.13 ([23, Thm 4.5]). *The morphism F is an equivalence of log stacks.*

4.5. Back to the big picture

We end by mentioning a type of converse to the philosophic principal mentioned at the beginning of this section. We have seen that since log smoothness

es-big-picture

includes degenerate objects, log geometry can naturally lead to compactifications; however, it is also generally true that we do not end up with “too many” degenerate objects.

Philosophy. Log geometry controls degenerations.

In higher dimensions, compactifications tend to have unwanted extra components. Log geometry helps to cut down on these components. Let us give some inkling of an idea as to why this should be true. Suppose \mathfrak{X} is an algebraic stack which is irreducible. Suppose we can find a proper algebraic stack $\tilde{\mathfrak{X}}$ with a fine log structure and an open immersion $i : \mathfrak{X} \rightarrow \tilde{\mathfrak{X}}$ such that \mathfrak{X} is the trivial locus of $\tilde{\mathfrak{X}}$.

As we now explain, log geometry provides us with a good method of trying to show that $\tilde{\mathfrak{X}}$ is irreducible as well. If k is separably closed and $x : \text{Spec } k \rightarrow \tilde{\mathfrak{X}}$ is a morphism, then pulling back the log structure on $\tilde{\mathfrak{X}}$ endows $\text{Spec } k$ with a fine log structure. By [24, Lemma 2.10], it follows that this is the log structure associated to a morphism of monoids $P \rightarrow k$, where P is fine. Hence, we have a strict closed immersion of log schemes $j : \text{Spec } k \rightarrow \text{Spec } k[[P]]$, where $\text{Spec } k[[P]]$ is given its canonical log structure. Note that the generic point $\text{Spec } K$ of $\text{Spec } k[[P]]$ carries the trivial log structure. Therefore, if x factors as a morphism of log stacks through j , then we automatically obtain a commutative diagram

$$\begin{array}{ccccc}
 \mathfrak{X} & \xrightarrow{i} & \tilde{\mathfrak{X}} & & \\
 \uparrow y & & \nearrow & & \uparrow x \\
 \text{Spec } K & \longrightarrow & \text{Spec } k[[P]] & \longleftarrow j & \text{Spec } k
 \end{array}$$

and hence x is the specialization of the point y of \mathfrak{X} . We see then that the log structure on $\text{Spec } k$ obtained from $\tilde{\mathfrak{X}}$ somehow serves as a compass telling us which way to look in order to find a family degenerating to our given point x of $\tilde{\mathfrak{X}}$.

5. D-semistability and log structures

D-SS

Convention. Throughout this chapter, every scheme is over an algebraically closed field. X will be a normal crossing variety, by which we mean a variety for which every closed point $x \in X$ has an étale neighborhood of the form $\text{Spec } \frac{k[x_1, \dots, x_n]}{(x_1 \cdots x_r)} \rightarrow X$ such that x corresponds to the point with coordinate $(0, \dots, 0)$. By a standard neighborhood of $x \in X$, we mean the étale neighborhood of x above. \mathcal{M}_k denotes the log structure of the standard log point $\mathbb{N} \rightarrow k, n \mapsto 0^n$.

5.1. Introduction

To study the geometry of a normal crossing variety X , e.g. to study the deformation theory of X , one would like to ask the following questions:

wording

- 1 (1) Can we embed X into another variety with codimension 1, i.e. an embedding $i : X \rightarrow \mathcal{X}$ as a normal crossing divisor?] fix wording
- 2 (2) Can we find a semi-stable smoothing of X , i.e. embed $X \rightarrow \text{Spec } k$ in a flat family over a curve $\mathcal{X} \rightarrow C$, s.t. there exists a diagram

$$\begin{array}{ccccc}
 X & \longrightarrow & \mathcal{X} & \longleftarrow & \mathcal{X}^* = \mathcal{X} \setminus X \\
 \downarrow & & \downarrow & & \downarrow f^* \\
 \text{Spec } k & \xrightarrow{0} & C & \longleftarrow & C^* = C \setminus 0
 \end{array}$$

where \mathcal{X} is smooth, the squares are cartesian and f^* is smooth? (If the answer is yes, then we say X is semi-stably smoothable.) ?

The answer to these questions are not always yes, because the existence of such maps would imply the existence of certain log structures on X , which in turn would imply intrinsic condition on X , so their existence is not guaranteed.

emb *Example 5.1* (log structure of embedding type). If we can find an embedding as in 1 above, then $i^*(j_* \mathcal{O}_{\mathcal{X}^*}^\times) \rightarrow \mathcal{O}_X$ defines a log structure \mathcal{M}_X on X , which étale locally has a chart $\mathbb{N}^r \rightarrow \text{Spec } \frac{k[x_1, \dots, x_n]}{(x_1 \cdots x_r)}$ sending the element e_i in the standard basis of \mathbb{N}^r to x_i , making X a log smooth variety over $\text{Spec } k$ with the trivial log structure. This is called a log structure of embedding type.

funct *Remark 5.2*. This construction is functorial in X .

sm *Example 5.3* (log structure of semi-stable type). If we can find a semi-stable smoothing as in 2 above, then by remark 5.2, what we have in this case is not only a log structure of embedding type on X , but also a morphism (of sheaf of monoids) $f^b : f^* \mathcal{M}_k \rightarrow \mathcal{M}_X$, which makes X a log smooth variety over $(\text{Spec } k, \mathcal{M}_k)$. Étale locally a chart for the log structure on X can be put in the form $(\text{Spec } \frac{k[x_1, \dots, x_n]}{(x_1 \cdots x_r)}, \mathbb{N}^r, e_i \mapsto x_i, i = 1, \dots, r)$. Modulo the units in the monoid, the morphism of quotient monoids induced by $f^b : f^* \mathcal{M}_k \rightarrow \mathcal{M}_X$ is just the diagonal $\Delta : \mathbb{N} \rightarrow \mathbb{N}^r$. Such a pair $(\mathcal{M}_X, f^b : f^* \mathcal{M}_k \rightarrow \mathcal{M}_X)$ is called a log structure of semi-stable type on X .

Remark 5.4. The log structure \mathcal{M}_k on $\text{Spec } k$ can be defined by the pullback of the log structure $(\mathcal{M}_0 = j_* \mathcal{O}_C^\times \rightarrow \mathcal{O}_C)$ on C , i.e. the log structure defined by the divisor 0 of C . We have an isomorphism $\overline{\mathcal{M}}_k := \mathcal{M}_k / k^\times \cong (\mathcal{M}_0 / \mathcal{O}_C^\times)_0 \cong \mathbb{N}$, where the second isomorphism assigns each function to its vanishing order at the point 0 in \mathbb{N} . This gives a geometric interpretation of the standard log point.

Concerning the existence of such log structure on the normal crossing variety X , we have the following theorems ([22], Sec.11):

spacing

what construction?

where \mathcal{M}_k is hollow log str.

but it depends on choice of uniformizer

embed

Theorem 5.5. Let X be a normal crossing variety over the spectrum of an algebraically closed field, then X can be equipped with a log structure of embedding type iff there exists a line bundle \mathcal{L} on X such that

$$\mathcal{E}xt_X^1(\Omega_X^1, \mathcal{O}_X)|_D \cong \mathcal{L}|_D$$

where D is the non-smooth locus of X .

Remark 5.6. It is not hard to see $\mathcal{E}xt_X^1(\Omega_X^1, \mathcal{O}_X)|_D$ is a line bundle on D .

Definition 5.7. Let X and D as before. If $\mathcal{E}xt_X^1(\Omega_X^1, \mathcal{O}_X)|_D$ is a trivial on D , then we say X is d-semistable.

d-semi

Theorem 5.8 (d-semistability). Let X be a normal crossing variety over the spectrum of an algebraically closed field, then X can be equipped with a log structure over the standard log point, such that the structure morphism is log smooth if and only if X is d-semistable.

Generalization of these theorems can be found in [43], section 3.

Corollary 5.9. If X has a smoothing, then X is d-semistable.

In fact, we can put a log structure on X such that it is log smooth over the standard log point by Example 5.3, so we can apply Theorem 5.8

Remark 5.10. Being d-semistable is not equivalent to being semi-stably smoothable. See [49], 3, for counter examples.

tetra

Example 5.11 (a normal crossing variety that is not d-semistable[9]). Let X be the subvariety of \mathbb{P}^3 defined by the product of 4 linear equations $f = L_1 L_2 L_3 L_4 = 0$, it is of normal crossing, provided the four planes has no points in common. Then D is defined by the homogeneous ideal $(L_i L_j | 1 \leq i < j \leq 4)$, and it is not hard to calculate that $\mathcal{E}xt_X^1(\Omega_X^1, \mathcal{O}_X)|_D \cong \mathcal{O}_D(4)$, which is not trivial. So X is not d-semistable.

Corollary 5.12. The X in Example 5.11 is not semi-stably smoothable.

Remark 5.13. If we put X in the 1-dimensional family \mathcal{X} defined by $f + tg = 0$ with parameter t , where g is a smooth quartic, then X is the fiber over $t = 0$ and for generic g , \mathcal{X} over $(t \neq 0)$ is smooth. But the whole space of this family is not smooth: in fact, for a generic g , this family is singular at the 24 points of $D \cap \{g = 0\}$. However, a single blowing up at such a point gives a $\mathbb{P}^1 \times \mathbb{P}^1$. Contract along either ruling gives back a family with parameter t which has one less singularity. If we do this process to all 24 points, we will get a family which is a semi-stable smoothing of \tilde{X} , the blowing-up of X at those 24 points. \tilde{X} is d-semistable. For details, see [9], Remark (1.14).

give ref for this def.

defn?

fix citation

wording

Fix notation
...

5.2. Refined analysis of the existence of log structures

Let's analyze the situation. we want to break the job of finding a suitable log structure over the standard log point into 2 steps:

- (1) Put a log structure of embedding type on X .
- (2) See if it is possible to make that log structure semi-stable.

We will see that two related obstructions arise naturally, where the vanishing of the first corresponds to the first step, and the vanishing of the second, which means precisely being d-semistable, allows us to do the second step.

Since étale locally a log structure of embedding type always exists, let's consider the stack \mathcal{G} , which to each $U \in X_{\text{ét}}$ associates the groupoid of log structure of embedding type on U . Then it is not hard to show (using Artin's approximation theorem) that any two element of U is locally isomorphic, which means \mathcal{G} is a gerbe. Since $\text{Aut } \mathcal{G} \cong \mathcal{K}$, where \mathcal{K} is the kernel of the restriction map $\mathcal{O}_X^\times \rightarrow \mathcal{O}_D^\times$, we have:

I think this means it is hard.

Proposition 5.14. *There is an obstruction η in $H^2(X_{\text{ét}}, \mathcal{K})$ whose vanishing is equivalent to the existence of a log structure of embedding type on X .*

For the calculation of this obstruction, we state the following result (See [22], Sec.11 and [43], Sec.3):

gerbe **Proposition 5.15.** *In the long exact sequence of cohomology associated to the short exact sequence $1 \rightarrow \mathcal{K} \rightarrow \mathcal{O}_X^\times \rightarrow \mathcal{O}_D^\times \rightarrow 1$, the line bundle $\text{Ext}_X^1(\Omega_X^1, \mathcal{O}_X)|_D$ maps to $-\eta \in H^2(X_{\text{ét}}, \mathcal{K})$*

Combining these results with the exactness of the long exact sequence, we get Theorem 5.5.

gr **Remark 5.16.** General theory tells us if $\eta = 0$, then the set of all log structure of embedding type on X is naturally a torsor under $H^1(X_{\text{ét}}, \mathcal{K})$. In general the set of all log structure of embedding type is only a pseudo torsor under this group.

Suppose $\eta = 0$, then we can put a log structure of embedding type on X , which maps to $(\text{Spec } k, k^\times)$. To go one step further, i.e., to make it a log structure of semistable type, we need a morphism of monoids $f^* \mathcal{M}_k \rightarrow \mathcal{M}_X$, such that in an standard neighborhood of $x \in X$, a chart of the morphism is given by the diagonal $\Delta : \mathbb{N} \rightarrow \mathbb{N}^r$, where r is the number of irreducible components passing through x .

Since $\mathcal{M}_k \cong \mathbb{N} \oplus k^\times$ (non-canonically), and the image of $k^\times \subset \mathcal{M}_k$ is determined by the underlying morphism of schemes. To give the morphism wanted from \mathcal{M}_k to \mathcal{M}_X , we only have to specify the image of an element in \mathcal{M}_k having vanishing order 1.

Now the question becomes a lifting problem for morphism of sheaves in monoids:

$$\begin{array}{ccc}
 & & \mathcal{M}_X \\
 & \nearrow & \downarrow \beta \\
 \mathbb{N} \cong f^{-1}\mathcal{M}_k & \xrightarrow{\bar{f}^b} & \mathcal{M}_X
 \end{array}$$

$\bar{f}^b(\alpha)$

where étale locally \bar{f}^b is the diagonal Δ . To lift it is equivalent to lift $\Delta(1)$. Consider the sheaf of all the possible local liftings of $\Delta(1)$, $T = \beta^{-1}(\Delta(1))$, then T is a torsor under \mathcal{O}_X^\times . To find a lifting of $\Delta(1)$ is equivalent to find a global section of T , i.e. a trivialization of T .

It seems like we've got an obstruction of finding a log structure on X which is semi-stable in $H^1(X_{\text{ét}}, \mathcal{O}_X) = \text{Pic}(X)$. This is, however, not quite true. In fact, what we got is for each log scheme X with a log structure of embedding type, the obstruction of making it semi-stable. And our original question (on the existence of log structure of semi-stable type) allows some ambiguity of choosing the log structure of embedding type \mathcal{M}_X on X . As we said in remark 5.16, in this case the set of all log structure of embedding type on X is an $H^1(X_{\text{ét}}, \mathcal{K})$ -torsor, which implies:

Proposition 5.17. *If $\eta = 0$, i.e. there exists a log structure of embedding type on X . Then there is an obstruction for finding a log structure of semi-stable type on X , $\eta' \in H^1(X_{\text{ét}}, \mathcal{O}_X^\times)/H^1(X_{\text{ét}}, \mathcal{K})$, whose vanishing is equivalent to the existence of such a log structure.*

By the long exact sequence of cohomology, $H^1(X_{\text{ét}}, \mathcal{O}_X^\times)/H^1(X_{\text{ét}}, \mathcal{K})$ embeds into $H^1(D_{\text{ét}}, \mathcal{O}_D^\times) \cong \text{Pic}(D)$. For the calculation of η' as an element of $\text{Pic}(D)$, we state the following proposition (See [43]).

Proposition 5.18. *We have $-\eta' = [\mathcal{E}xt_X^1(\Omega_X^1, \mathcal{O}_X)]_D \in \text{Pic}(D)$.*

Combining these two proposition, we get Theorem 5.8.

6. Stacks of logarithmic structures

LogStacks

6.1. A motivating example

Before introducing the stack LOG_S classifying fine log structures on schemes over a fine log scheme S , constructed by Olsson in [42], let us look at an example, which will give the local covers of LOG_S .

DF-log-str

Definition 6.1. Let X be a scheme and $r \geq 1$ an integer. A *Deligne-Faltings log structure of rank r on X* (abbreviated as a *DF log structure of rank r*) is the following data:

- a sequence L_1, \dots, L_r of line bundles on X , and

- a morphism $s_i : L_i \rightarrow \mathcal{O}_X$ of line bundles, for each i .

Consider the following three categories fibered in groupoids over the category of schemes:

- (1) the category of triples $(\underline{X}, L, s : L \rightarrow \mathcal{O}_{\underline{X}})$ consisting of a scheme \underline{X} and a DF log structure of rank 1 on \underline{X} ;
- (2) the category of pairs $(X, \beta : \mathbb{N} \rightarrow \overline{\mathcal{M}}_X)$ consisting of a fine log scheme X and a morphism of sheaves of monoids β that étale locally lifts to a chart: $\tilde{\beta} : \mathbb{N} \rightarrow \mathcal{M}_X$;
- (3) the quotient stack $[\mathbb{A}^1/\mathbb{G}_m]$, where the quotient is formed with respect to the multiplication action of \mathbb{G}_m on \mathbb{A}^1 .

~~The description of morphisms in these fibered categories and their structural functors to the category of schemes are evident.~~

*If evident
no reason to
state*

Lemma 6.2. ([24], complement 1). *These three categories fibered in groupoids are equivalent.*

Let us sketch the proof. Given a DF log structure $(L, s : L \rightarrow \mathcal{O}_{\underline{X}})$ of rank 1 on \underline{X} , define a sheaf of monoids \mathcal{M}' on \underline{X} to be

$$\coprod_{n \geq 0} \text{Isom}(\mathcal{O}_{\underline{X}}, L^{\otimes n}),$$

the sheafification of the presheaf that takes U to $\coprod_{n \geq 0} \text{Isom}(\mathcal{O}_U, (L|_U)^{\otimes n})$. It comes with a natural morphism of sheaves of monoids $\mathcal{M}' \rightarrow \mathbb{N}$, where the monoid structure on \mathcal{M}' is induced by

$$(n, a : \mathcal{O} \rightarrow L^{\otimes n}) \cdot (m, b : \mathcal{O} \rightarrow L^{\otimes m}) = (n + m, a \otimes b).$$

The map $s : L \rightarrow \mathcal{O}_{\underline{X}}$ induces a morphism

$$\text{Isom}(\mathcal{O}_{\underline{X}}, L^{\otimes n}) \xrightarrow{\otimes s} \text{Hom}(\mathcal{O}_{\underline{X}}, \mathcal{O}_{\underline{X}}) = \mathcal{O}_{\underline{X}}$$

of sheaves, hence giving a pre-log structure on \mathcal{M}' :

$$\mathcal{M}' \rightarrow \mathcal{O}_{\underline{X}}.$$

We take \mathcal{M}_X to be the log structure associated to this pre-log structure \mathcal{M}' . Note that $\mathcal{M}'/\mathcal{O}_X^* \cong \mathbb{N}$, and we define $\beta : \mathbb{N} \rightarrow \overline{\mathcal{M}}_X$ to be the composite

$$\beta : \mathbb{N} \cong \mathcal{M}'/\mathcal{O}_X^* \rightarrow \mathcal{M}_X/\mathcal{O}_X^*.$$

Locally the line bundle L is trivial, and one can choose a trivialization of L , which gives trivializations of all $L^{\otimes n}$. Sending $n \in \mathbb{N}$ to this trivialization defines a section $\mathbb{N} \rightarrow \mathcal{M}'$, and hence a section $\tilde{\beta} : \mathbb{N} \rightarrow \mathcal{M}' \rightarrow \mathcal{M}_X$. One can check that this is a chart.

Conversely, given a fine log structure $(\mathcal{M}_X, \mathcal{M}_X \xrightarrow{\alpha} \mathcal{O}_X)$ on \underline{X} with a morphism $\beta : \mathbb{N} \rightarrow \overline{\mathcal{M}}_X$ that étale locally lifts to a chart $\tilde{\beta} : \mathbb{N} \rightarrow \mathcal{M}_X$, we have a

section $\beta(1)$ of $\overline{\mathcal{M}}_X$, and its inverse image under $\pi : \mathcal{M}_X \rightarrow \overline{\mathcal{M}}_X$ is an \mathcal{O}_X^* -torsor, which corresponds to a line bundle L . The composition

$$\pi^{-1}(\beta(1)) \subset \mathcal{M}_X \xrightarrow{\alpha} \mathcal{O}_X$$

gives a morphism of line bundles $s : L \rightarrow \mathcal{O}_X$.

Giving a morphism $\underline{X} \rightarrow [\mathbb{A}^1/\mathbb{G}_m]$ is equivalent to giving a \mathbb{G}_m -torsor (namely a line bundle L) with a \mathbb{G}_m -equivariant morphism to \mathbb{A}^1 :

$$\begin{array}{ccc} Y & \longrightarrow & \mathbb{A}^1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]. \end{array}$$

This diagram is equivalent to the following one

$$\begin{array}{ccc} Y & \xrightarrow{s} & \mathbb{A}_X^1 \\ \downarrow & & \downarrow \\ X & \longrightarrow & [\mathbb{A}^1/\mathbb{G}_m]_X, \end{array}$$

and the top arrow is \mathbb{G}_m -equivariant, namely a morphism of line bundles $s : L \rightarrow \mathcal{O}_X$. This finishes the proof.

In fact, in the three fibered categories, one can replace \mathbb{N} by \mathbb{N}^r , and rank 1 DF-log structure by rank r DF-log structure, and replace $[\mathbb{A}^1/\mathbb{G}_m]$ by $[\mathbb{A}^r/\mathbb{G}_m^r]$ (which is equivalent to $[\mathbb{A}^1/\mathbb{G}_m]^r$), and they are still equivalent.

More generally, let P be a fine monoid and S a scheme, and let $S[P]$ be the product $S \times_{\text{Spec } \mathbb{Z}} \text{Spec } \mathbb{Z}[P]$, which has a fine log structure coming from the chart $P \rightarrow \mathbb{Z}[P]$ (2.8). For an affine S -scheme $\text{Spec } R$, the set of R -points $S[P](R)$ is the set of monoid homomorphisms $\text{Hom}_{\text{mon}}(P, R)$, where R is regarded as a multiplicative monoid. Let P^{gp} be the group associated to P . For any affine S -scheme $\text{Spec } R$, the group $\text{Hom}_{\text{mon}}(P^{\text{gp}}, R) = \text{Hom}_{\text{gp}}(P^{\text{gp}}, R^*)$ acts on the set $\text{Hom}_{\text{mon}}(P, R)$ by pointwise multiplication. This induces an action of the S -group scheme $S[P^{\text{gp}}]$ on $S[P]$. When $S = \text{Spec } k$ for a field k and P is saturated and torsion-free, the k -group variety $S[P^{\text{gp}}]$ is a torus, and $S[P]$ is a toric variety with respect to this torus action.

We have the following.

toric-stack

Lemma 6.3. ([42], 5.14, 5.15) *The following two categories fibered in groupoids over the category of S -schemes are equivalent:*

- (1) *the category of pairs $(X, \beta : P \rightarrow \overline{\mathcal{M}}_X)$ consisting of a fine log scheme X with a morphism $\underline{X} \rightarrow S$ and a morphism of sheaves of monoids β that fppf locally lifts to a chart: $\tilde{\beta} : P \rightarrow \mathcal{M}_X$;*
- (2) *the quotient stack $\mathcal{S}_P := [S[P]/S[P^{\text{gp}}]]$.*

If in addition P is fs, then one can replace “fppf” by “étale”.

refer back to
earlier discussion
of log stacks
on stacks

In fact, the action of $S[P^{\text{gp}}]$ on $S[P]$ extends to an action on the log structure on $S[P]$, and so this log structure descends to a log structure \mathcal{M}_{S_P} on the stack S_P , and there is a natural morphism $\pi_P : P \rightarrow \overline{\mathcal{M}}_{S_P}$ of sheaves of monoids that fppf locally lifts to a chart. This is the universal pair $(\mathcal{M}_{S_P}, \pi_P)$ on S_P that induces the equivalence in (6.3) above.

ty-toric-stack

6.1.1. Moreover, for a morphism $h : Q \rightarrow P$ of fine monoids, the induced morphism

$$S[h] : S[P] \rightarrow S[Q]$$

is compatible with the actions of $S[P^{\text{gp}}], S[Q^{\text{gp}}]$ and the homomorphism $S[h^{\text{gp}}] : S[P^{\text{gp}}] \rightarrow S[Q^{\text{gp}}]$, hence it descends to a morphism

$$S(h) : S_P \rightarrow S_Q$$

of S -stacks. The map $h : Q \rightarrow P$, regarded as a morphism of constant sheaves, induces a morphism $S(h)^* \mathcal{M}_{S_Q} \rightarrow \mathcal{M}_{S_P}$ of log structures, making $S(h)$ into a morphism of S -log stacks.

6.2. The stack of log structures

Now we can discuss the stack LOG_S parameterizing fine log structures.

Let S be a fine log scheme. Define LOG_S to be the category with

- objects: morphisms $X \rightarrow S$ of fine log schemes, and
- morphisms: strict morphisms $X \rightarrow Y$ over S .

With the functor $(X \rightarrow S) \mapsto (\underline{X} \rightarrow \underline{S})$ from $LOG_S \rightarrow \text{Sch}_{\underline{S}}$, this defines a fibered category over \underline{S} . One of the main results in [42] is the following.

Theorem 6.4. ([42], 1.1) LOG_S is an algebraic stack locally of finite presentation over \underline{S} .

Here for an algebraic stack we use a slightly different definition from [32, 4.1]. Namely the first axiom there that the diagonal is representable, separated and quasi-compact, is replaced by that the diagonal is representable and of finite presentation. In fact the stack LOG_S is not quasi-separated [42, 3.17].

Here are two basic properties of LOG_S .

Proposition 6.5. ([42], 3.19, 3.20) (1). The natural map $i_S : \underline{S} \rightarrow LOG_S$ corresponding to the identity morphism $S \rightarrow S$ is an open immersion;

(2). The 2-functor

$$S \mapsto LOG_S : \{\text{fine log schemes}\} \rightarrow \{\text{algebraic stacks}\}$$

preserves fiber product. More precisely, if

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ S' & \longrightarrow & S \end{array}$$

is a Cartesian square of fine log schemes, then the induced diagram

$$\begin{array}{ccc} \text{LOG}_{X'} & \longrightarrow & \text{LOG}_X \\ \downarrow & & \downarrow \\ \text{LOG}_{S'} & \longrightarrow & \text{LOG}_S \end{array}$$

is a 2-Cartesian square of algebraic stacks.

6.3. What is LOG_S good for?

One can use this stack LOG_S to reinterpret many concepts in log geometry. Note that for a morphism $f : X \rightarrow S$ of fine log schemes, the induced morphism $\text{LOG}(f) : \text{LOG}_X \rightarrow \text{LOG}_S$ is faithful, hence representable.

def-LOG(f)

Definition 6.6. Let P be a property of representable morphisms of algebraic stacks. Then we say that $f : X \rightarrow S$ has property $\text{LOG}(P)$ if $\text{LOG}(f) : \text{LOG}_X \rightarrow \text{LOG}_S$ has property P . We say that f has property weak $\text{LOG}(P)$ if the map $\underline{X} \rightarrow \text{LOG}_S$ corresponding to the given morphism $f : X \rightarrow S$ has property P .

6.3.1. Caution: The diagram

$$\begin{array}{ccc} \underline{X} & \xrightarrow{i_X} & \text{LOG}_X \\ \downarrow f & & \downarrow \text{LOG}(f) \\ \underline{S} & \xrightarrow{i_S} & \text{LOG}_S \end{array}$$

does not necessarily commute. It commutes if and only if f is strict.

Recall from (3.10) the notion of log smoothness and log étaleness.

Th-LOG-sm

Theorem 6.7. For a morphism $f : X \rightarrow S$ of fine log schemes, f is LOG smooth (resp. LOG étale) if and only if f is log smooth (resp. log étale), if and only if f is weakly LOG smooth (resp. weakly LOG étale).

This is part of [42, 4.6].

6.3.2. Another application is the following. Consider the deformation problem for a log smooth integral morphism $f_0 : X_0 \rightarrow B_0$ of fine log schemes and a strict square-zero thickening $B_0 \rightarrow B$ defined by an ideal $J \subset \mathcal{O}_B$. We see in (3, 3.20) that the solution to this deformation problem is the cohomology groups of the log tangent bundle T_{X_0/B_0} . The stack LOG_S provides another way of thinking of this problem.

wording

fix citation

wording

By (6.7), the log smooth morphism $f_0 : X_0 \rightarrow B_0$ induces a representable smooth morphism $\underline{X}_0 \rightarrow \text{LOG}_{B_0}$, also denoted f_0 , and the deformation problem

$$\begin{array}{ccc} X_0 & \hookrightarrow & X \\ \downarrow & & \downarrow \\ B_0 & \hookrightarrow & B \end{array}$$

is equivalent to the following

$$\begin{array}{ccc} \underline{X}_0 & \dashrightarrow & \underline{X} \\ f_0 \downarrow & & \downarrow f \\ \text{LOG}_{B_0} & \longrightarrow & \text{LOG}_B \end{array}$$

The solution to this deformation problem is the cohomology groups of the *ordinary tangent bundle* $T_{\underline{X}_0/\text{LOG}_{B_0}}$, therefore, (3, 3.20) holds with T_{X_0/B_0} replaced by $T_{\underline{X}_0/\text{LOG}_{B_0}}$. In fact, we have $\Omega_{X_0/B_0}^1 \cong \Omega_{\underline{X}_0/\text{LOG}_{B_0}}$ (cf. ([45], 3.8)).

See section 7 for the general log deformation.

wondering

ction-loc-stru

6.4. Local structure of LOG_S .

For a fine log scheme S , the relation between the quotient stacks \mathcal{S}_P and LOG_S is that, the stack LOG_S can be covered by the relative versions of the \mathcal{S}_P 's.

Let $u : U \rightarrow S$ be a strict morphism of fine log schemes, such that the underlying morphism \underline{u} is étale. We will just say that u is an étale strict morphism, if there is no confusion. Let $\beta : Q \rightarrow \mathcal{M}_U$ be a chart, and let $h : Q \rightarrow P$ be a morphism of fine monoids. The chart β induces a strict morphism $U \rightarrow \underline{S}[Q]$, which we also denote by β .

Let \mathcal{S}_P be the quotient stack $[\mathcal{S}[P]/\mathcal{S}[P^{\text{gp}}]]$ with the natural fine log structure $\mathcal{M}_{\mathcal{S}_P}$ discussed before, and let \underline{S}_P be the underlying stack. Consider the 2-commutative diagram

$$\begin{array}{ccc} \underline{S}_P \times_{\underline{S}_Q} U & \xrightarrow{pr_2} & U \xrightarrow{\beta} \underline{S}[Q] \\ pr_1 \downarrow & & \downarrow \pi \\ \underline{S}_P & \xrightarrow{\underline{S}(h)} & \underline{S}_Q \end{array}$$

Let Z be the \underline{S} -log stack with underlying stack $\underline{Z} = \underline{S}_P \times_{\underline{S}_Q} U$ and the inverse image log structure $\mathcal{M}_Z = pr_1^* \mathcal{M}_{\mathcal{S}_P}$. Applying pr_1^* to the morphism of log structures $\underline{S}(h)^* \mathcal{M}_{\underline{S}_Q} \rightarrow \mathcal{M}_{\underline{S}_P}$ (cf. (6.1.1)), noting that $\pi \circ \beta : U \rightarrow \underline{S}_Q$ is strict, we obtain a morphism $pr_2^* \mathcal{M}_U \rightarrow \mathcal{M}_Z$, making pr_2 into a morphism of log stacks $pr_2 : Z \rightarrow U$. This gives a morphism

$$\underline{Z} \rightarrow \text{LOG}_U \rightarrow \text{LOG}_S.$$

ref.

\underline{U} ?

Prop-loc-stru

Proposition 6.8. ([42], 5.25) For any fine log scheme S , the natural morphism

$$\coprod_{(U, \beta, h)} S_P \times_{S_Q} U \rightarrow LOG_S$$

is a representable étale surjection, where the disjoint union is taken over the isomorphism classes of all triples (U, β, h) consisting of an étale strict morphism $U \rightarrow S$, a chart $Q \xrightarrow{\beta} \mathcal{M}_U$, and a morphism $h : Q \rightarrow P$ of fine monoids, for some fine monoids P and Q .

LogDef

7. Log deformation theory in general

As we know, for schemes, the general deformation theory is not as easy as the smooth case. To understand deformation theory of general morphisms, one has to use the full power of the cotangent complex. (see [17][18]) In log geometry, one can generalize it to get a reasonable theory of logarithmic cotangent complex.

spring

This log cotangent complex will be compatible with the usual cotangent complex when the morphism in question is strict and is also compatible with log smooth deformation theory for log smooth morphisms. Basically, this is an application of the deformation theory of representable morphisms to algebraic stacks ([46]) to the classifying morphisms from the underlying scheme of X to the stack LOG_Y ([42], see also section 6 of this chapter.)

Convention. We will focus on the category of fine log schemes. For a log scheme X , \underline{X} means the underlying scheme of X .

Remark 7.1. We will work with the category $D'(\underline{X}_{\text{ét}})$ and similar categories, and one can talk about distinguished triangles and Ext's in these categories. For relevant definitions, see [45] and [46].

Our presentation here follows [45].

Remark that another approach to cotangent complex due to Bublitz

7.1. The Log Cotangent Complex

In Section 6, an Artin stack LOG_Y is defined so that to give a morphism of log schemes $X \rightarrow Y$ is equivalent to give a representable morphism $\underline{X} \rightarrow LOG_Y$. Thus one may think deformations of the morphism of log schemes $X \rightarrow Y$ as deformations of the representable morphism from \underline{X} to the stack LOG_Y . In [46], the deformation theory of such morphisms was studied in detail. As an application of this theory, one makes the following definition:

wonderful

Definition 7.2. For a morphism of log schemes $f : X \rightarrow Y$, the logarithmic cotangent complex of f is defined by $L_f = L_{\underline{X}/LOG_Y}$, where the right hand side is the cotangent complex of the morphism $\underline{X} \rightarrow LOG_Y$ defined in Section 6.

Remark 7.3. One should think about L_f as an object of the category $D'_{qcoh}(\underline{X}_{ét})$. In the above definition, the right hand side is an object of the category $D'_{qcoh}(\underline{X}_{lis-ét})$. As the restriction functor $D'_{qcoh}(\underline{X}_{lis-ét}) \rightarrow D'_{qcoh}(\underline{X}_{ét})$ is an equivalence of categories, no information of the cotangent complex would be lost.

7.2. Basic Properties

For every morphism of fine log schemes $f : X \rightarrow Y$ the log cotangent complex is a projective system

$$L_f = (\dots \rightarrow L_f^{\geq -n-1} \rightarrow L_f^{\geq -n} \rightarrow \dots \rightarrow L_f^{\geq 0})$$

where each $L_f^{\geq -n}$ is an essentially constant ind-object in $D^{[-n,0]}(\mathcal{O}_X)$ (The derived category of \mathcal{O}_X -modules supported in $[-n, 0]$).

The log cotangent complex L_f has the following properties:

- (1) For any $n \geq 0$, the natural map $\tau_{\geq -n} L_f^{\geq -n-1} \rightarrow L_f^{\geq -n}$ is an isomorphism.
- (2) If f is strict, then the system $(\tau_{\geq -n} L_{f'})$ represents L_f , where $L_{f'}$ is the usual cotangent complex of the underlying morphism of schemes f' .
- (3) If $f : X \rightarrow Y$ is log smooth, then the sheaf of log differentials $\Omega_{X/Y}^1$ represents L_f .

logsm
split

- (4) If

$$\begin{array}{ccc} X' & \xrightarrow{a} & X \\ g \downarrow & & \downarrow f \\ Y' & \xrightarrow{b} & Y \end{array}$$

is a commutative diagram of fine log schemes, then there is a natural map

$$a^* L_f \rightarrow L_g$$

which is an isomorphism if the square above is cartesian and f is log flat. Furthermore, if the composite $X' \rightarrow Y' \rightarrow Y$ satisfies the condition (T) below, then the map

$$g^* L_b \oplus a^* L_f \rightarrow L_{bg}$$

is also an isomorphism.

triangle

- (5) Given a composite

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

satisfying condition (T) below, there is a natural map

$$L_f \rightarrow f^* L_g[1]$$

making the resulting triangle

tri

(7.4) $f^* L_g \rightarrow L_{gf} \rightarrow L_f \rightarrow f^* L_g[1]$

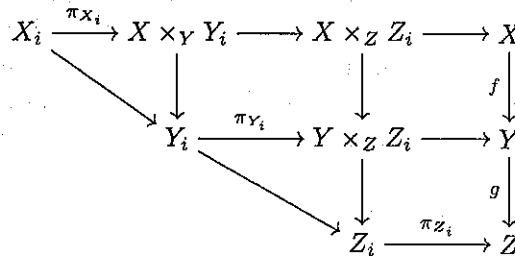
distinguished.

Remark 7.5. In resplit and retriangle above, f^*, a^*, g^* should be understood in the derived sense.

Remark 7.6. One might hope a theory of log cotangent complex in which every triangle (7.4) is distinguished. However, such a theory is too good to exist.

On the other hand, Gabber has shown in [45] that if one loosen the requirement 3, then one can obtain a theory of log cotangent complexes for which one has a distinguished triangle (7.4) for all composites $X \xrightarrow{f} Y \xrightarrow{g} Z$.

The Condition (T) mentioned above is the following:
There exists a family of commutative diagrams



such that

- (1) The underlying schemes of X_i, Y_i, Z_i are all affine.
- (2) The π 's are all strict, and their underlying morphisms are flat and locally of finite presentation.
- (3) The underlying family of morphisms of schemes of $\{X_i \rightarrow X\}$ is jointly surjective.
- (4) There exists charts

$$\beta_{X_i} : Q_{X_i} \rightarrow \mathcal{M}_{X_i}, \beta_{Y_i} : Q_{Y_i} \rightarrow \mathcal{M}_{Y_i}, \beta_{Z_i} : Q_{Z_i} \rightarrow \mathcal{M}_{Z_i}$$

and injective maps

$$Q_{Z_i} \rightarrow Q_{Y_i} \rightarrow Q_{X_i}$$

compatible with the morphisms f_i, g_i and

$$\text{Tor}_{\mathcal{O}_{Z_i} \otimes_{\mathbb{Z}[Q_{Z_i}]} \mathbb{Z}[Q_{Y_i}]}^j(\mathcal{O}_{Z_i} \otimes_{\mathbb{Z}[Q_{Z_i}]} \mathbb{Z}[Q_{X_i}], \mathcal{O}_{Y_i}[G]) = 0 \text{ for all } j > 0.$$

Here $G := \text{Coker}(Q_{Z_i}^{gp} \rightarrow Q_{Y_i}^{gp})$ and $\mathcal{O}_{Y_i}[G]$ is viewed as an $\mathcal{O}_{Z_i} \otimes_{\mathbb{Z}[Q_{Z_i}]} \mathbb{Z}[Q_{Y_i}]$ -algebra via the map

$$\mathcal{O}_{Z_i} \otimes_{\mathbb{Z}[Q_{Z_i}]} \mathbb{Z}[Q_{Y_i}] \rightarrow \mathcal{O}_{Y_i}[G], t \otimes e_q \mapsto g_i^*(t) \beta_{Q_{Y_i}}(q) \cdot \bar{q}$$

where \bar{q} denotes the image of q in G .

See
be precise

7.3. Deformation Theory of Log Schemes in General

In this section, we explain the relation between the log cotangent complex and deformation theory of log schemes. Let $f : X \rightarrow Y$ be a morphism of fine log schemes and let I be a quasi-coherent sheaf on X . Define a Y -extension of X by I to be a commutative diagram of log schemes

$$\begin{array}{ccc} X & \xrightarrow{j} & X' \\ f \downarrow & & \searrow f' \\ & & Y \end{array}$$

where j is a strict closed immersion ~~(meaning that the log structure on X is isomorphic to the pull-back of that of X')~~ defined by a square-zero ideal, together with an isomorphism $\epsilon_j : I \cong \text{Ker}(\mathcal{O}_{X'} \rightarrow \mathcal{O}_X)$. The set of Y extensions by I forms in a natural way a category $\text{Exal}_Y(X, I)$. Let $\text{Exal}_Y(X, I)$ be the set of isomorphism classes of this category.

There is a tautological equivalence of categories (see [46] for the meaning of the right hand side):

$$\text{Exal}_Y(X, I) \cong \text{Exal}_{\text{LOG}_Y}(X, I).$$

Hence by ([46], 1.1) and our definition of L_f we obtain the following result:

Theorem 7.7 *There is a natural bijection*

$$\text{Exal}_Y(X, I) \cong \text{Ext}^1(L_f, I).$$

It is precisely the theorem above that guarantees that general deformation theory is controlled by our logarithmic cotangent complex.

Definition 7.8. Let $j_0 : Y_0 \hookrightarrow Y$ be a strict closed immersion of fine log schemes defined by a square-zero ideal $I \subset \mathcal{O}_Y$, and let $f_0 : X_0 \rightarrow Y_0$ be a log flat morphism (As defined in Section 6). A *log flat deformation of X_0 to Y* is a cartesian square

$$\begin{array}{ccc} X_0 & \xrightarrow{j} & X \\ f_0 \downarrow & & \downarrow f \\ Y_0 & \xrightarrow{j_0} & Y \end{array}$$

with f log flat.

insert ref.

make precise.

precise ref.

fix.

To give a log flat deformation as above is equivalent to give a 2-commutative diagram

$$\begin{array}{ccc} X_0 & \xrightarrow{j} & X \\ \mathcal{L}_{f_0} \downarrow & & \downarrow \mathcal{L}_f \\ LOG_{Y_0} & \xrightarrow{j_0} & LOG_Y \end{array}$$

with \mathcal{L}_f flat. Thus from ([46], 1.4) we obtain the following:

Theorem 7.9. *Let J denote the ideal of LOG_{Y_0} in LOG_Y . Then*

- (1) *There exists a canonical class $o \in \text{Ext}^2(L_{f_0}, \mathcal{L}_{f_0}^* J)$ whose vanishing is equivalent to the existence of a log flat deformation of X_0 to Y .*
- (2) *If $o = 0$, then the set of isomorphism classes of log flat deformations of X_0 to Y is naturally a torsor under $\text{Ext}^1(L_{f_0}, \mathcal{L}_{f_0}^* J)$.*
- (3) *The automorphism group of any log flat deformation of X_0 to Y is canonically isomorphic to $\text{Ext}^0(L_{f_0}, \mathcal{L}_{f_0}^* J)$.*

This theorem gives an answer to the question of general deformation theory of log schemes.

7.4. Deformations of morphisms

Given a commutative diagram of solid arrows

defmor

(7.10)

$$\begin{array}{ccccc} X_0 & \xrightarrow{i} & X & & \\ & \searrow f_0 & & \searrow f & \\ & & Y_0 & \xrightarrow{j} & Y \\ & \searrow h_0 & & \searrow h & \\ & & Z_0 & \xrightarrow{k} & Z \end{array}$$

wording

where i, j, k are strict closed immersion defined by square-zero ideal sheaves I, J, K living on X, Y, Z respectively. The question is to find a dotted arrow f fitting in the diagram. To nail down f , we need some more data.

The morphisms h and g induce morphisms $w : h_0^* K \rightarrow I$ and $v : g_0^* K \rightarrow J$. Assume given a morphism $u : f_0^* J \rightarrow I$ such that the composite

$$h_0^* K = f_0^* g_0^* K \rightarrow f_0^* J \rightarrow I$$

is equal to w .

What we want to find is an f fitting in the diagram (7.10) such that the morphism $f_0^* J \rightarrow I$ induced by f is equal to u . This can also be solved by the logarithmic cotangent complex.

Theorem 7.11. *In the situation above, assume in addition that u induces a map $\mathcal{L}_{f_0}^* J' \rightarrow I$, where J' is the ideal of LOG_{Y_0} in LOG_Y , then there is a canonical class $o \in \text{Ext}^1(f_0^* L_{Y_0/Z_0}, I)$ which vanishes iff there exists f fitting in the diagram (7.10) such that the morphism $f_0^* J \rightarrow I$ induced by f is equal to u . If $o = 0$, then the set of such maps f is a torsor under the group $\text{Ext}^0(f_0^* L_{Y_0/Z_0}, I)$.*

For a proof of this theorem, see [45], Theorem 5.9.

8. Rounding

Rounding

8.1. What is rounding?

The process of “rounding”, in its most basic form, produces a manifold *with corners* from a *smooth* analytic space with a normal crossings divisor. So the corners are not rounded but rather the opposite: they are created. On the other hand, these corners are rather round and shapely. Also, anybody who has seen the construction under any name and hears the name “rounding” immediately knows what this is about. Evidently then, even though “rounding” might be something of a misnomer, it is a very good name. The origin of the name seems to be in work of Kajiwara, Nakayama and Ogus [21, 39].

In various moduli problems, the rounding of the moduli space often has a more natural topological interpretation than the moduli space itself. A good example is the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{g,n}$ of marked nodal curves, whose boundary is a normal crossings divisor. The “interior” $\mathcal{M}_{g,n}$ can be described topologically as a quotient of Teichmuller space by the appropriate mapping class group. There is a natural generalization of Teichmuller space involving 2-manifolds decorated with circles, due to Harvey [16]; the analogous quotient yields a topological description of the *rounding* of $\overline{\mathcal{M}}_{g,n}$, rather than the moduli space $\overline{\mathcal{M}}_{g,n}$ itself.

Another example is that of twisted curves, discussed in section 11. A twisted curve is an algebraic stack with a log structure, so it is a bit exotic. But its rounding is a good old topological space. A similar example occurs in current work of one of us (Gillam): the relative Hilbert stack of a marked Riemann surface can be defined algebraically using Jun Li style expansions, but it is not representable (except in some trivial cases). However its rounding is a topological space (a manifold with corners, even) which is relatively easy to describe. A similar phenomenon occurs in many moduli problems involving expansions, discussed briefly below.

The topological preeminence of the rounding of moduli spaces might ultimately be traced back to the preference in topology for operations involving real codimension one subspaces (e.g. connected sum of manifolds) as opposed to the algebro-geometric preference for complex codimension one operations (e.g. pushout of two smooth varieties along a common divisor). From this point of view, one

might think of, say, log geometry as an attempt to speak algebraically about various “real codimension one” phenomena.

8.2. The oriented real blowup

The most general rounding operation is the *Kato–Nakayama logarithmic space* associated to a log analytic space [26]. In the basic example of a smooth analytic space with log structure from a normal crossings divisor, the Kato–Nakayama space can be described in terms of *oriented real blowup*, which is a relatively simple rounding operation that can be described as follows. Suppose X is a topological space, $\pi : L \rightarrow X$ is a complex line bundle, and $s : X \rightarrow L$ is a section of π . Locally on X we can choose a trivialization $(\pi, \phi) : L \rightarrow X \times \mathbb{C}$ and consider the subspace

$$B_{L,s,\phi} X := \left\{ l \in L : |\phi(l)| \cdot (\phi s \pi)(l) = \phi(l) \cdot |(\phi s \pi)(l)| \right\}$$

of L . A continuous function $u : X \rightarrow \mathbb{C}^*$ yields a new trivialization $(\pi, u \cdot \phi)$, where $(u \cdot \phi)(l) := (u\pi)(l)\phi(l)$. The key observation is that $B_{L,s,\phi} X = B_{L,s,u \cdot \phi} X$, so the subspace $B_{L,s,\phi} X$ is independent of the choice of ϕ , hence one can define a subspace $B_{L,s} X \subseteq L$ by defining it locally on X using a trivialization, then gluing the locally defined subspaces. From the local picture using a trivialization, it is clear that the subspace $B_{L,s} X$ contains the zero section and $L|_{Z(s)}$ (where $Z(s) \subseteq X$ is the zero locus of s) and is invariant under the $\mathbb{R}_{>0}$ action on L inherited from the full \mathbb{C}^* scaling action. We let $B_{L,s}^* X$ be the complement of the zero section in $B_{L,s} X$ and we call

$$\text{Blo}_{L,s} X := (B_{L,s}^* X) / \mathbb{R}_{>0}$$

the *oriented real blowup* of X along (L, s) .

The space $\text{Blo}_{L,s} X$ is a closed subspace of the oriented circle bundle $S^1 L := L^* / \mathbb{R}_{>0}$ associated to L and is, in particular, proper over X . The projection $\tau : \text{Blo}_{L,s} X \rightarrow X$ is an isomorphism away from $Z(s)$ and $\tau^{-1}(Z(s))$ is oriented circle bundle $S^1 L|_{Z(s)}$. The spaces $B_{L,s} X$ and $\text{Blo}_{L,s} X$ are natural under pulling back line bundles and sections.

If X is an analytic space and $D \subseteq X$ is a Cartier divisor, then D determines a line bundle $\mathcal{O}_X(D)$ together with a section s whose zero locus is D . In this situation, we will write $B_D X$, $\text{Blo}_D X$, etc. and speak of the *oriented real blowup of X along D* . The space $\text{Blo}_D X$ inherits a differentiable structure from its inclusion in $S^1 \mathcal{O}_X(D)$.

The basic example to keep in mind is the oriented real blowup $\text{Blo}_0 \mathbb{C}$ of the complex plane \mathbb{C} at the origin. The origin is the zero locus of the identity map

Id : $\mathbb{C} \rightarrow \mathbb{C}$, hence

$$\begin{aligned} \text{Blo}_0 \mathbb{C} &= \{(z, Z) \in \mathbb{C} \times \mathbb{C}^* : |z|Z = z|Z|\} / \mathbb{R}_{>0} \\ &= \{(z, Z) \in \mathbb{C} \times S^1 : |z|Z = z\} \\ &\cong \mathbb{R}_{\geq 0} \times S^1, \end{aligned}$$

where the last isomorphism from $\mathbb{R}_{\geq 0} \times S^1$ is given by $(\lambda, Z) \mapsto (\lambda Z, Z)$. Evidently $\text{Blo}_0 \mathbb{C}$ is a half-infinite annulus whose boundary S^1 is the exceptional locus of $\tau : \text{Blo}_0 \mathbb{C} \rightarrow \mathbb{C}$ (the fiber over the origin).

8.3. The Kato–Nakayama space

Let (X, \mathcal{M}_X) be a fine and saturated logarithmic analytic space.

def-Xlog

Definition 8.1. [[26, 1.2]] We define its *canonical rounding*, or *Kato–Nakayama space*, denoted X^{\log} , as the space whose points are pairs (x, F) where $x \in X$ and $F : \mathcal{M}_{X,x} \rightarrow S^1$ is a monoid homomorphism satisfying $F(u) = u(x)/|u(x)|$ for every $u \in \mathcal{O}_{X,x}^* \subseteq \mathcal{M}_{X,x}$.

This space can be easily given a topology. Let us describe the topology in the special case where \mathcal{M}_X is the canonical log structure associated to a Cartier divisor $D \subseteq X$, see Example 2.7. Locally on X we can find $f_1, \dots, f_n \in \mathcal{M}_X(X)$ which, together with the units, generate \mathcal{M}_X . The map

$$\begin{aligned} X^{\log} &\rightarrow X \times (S^1)^n \\ (x, F) &\mapsto (x, F(f_{1,x}), \dots, F(f_{n,x})) \end{aligned}$$

is then easily seen to be a monomorphism onto a closed subset of $X \times (S^1)^n$, so we give X^{\log} the subspace topology so that this is a closed embedding. Since one can check easily that this topology doesn't depend on the choice of generators f_1, \dots, f_n , the locally defined topologies glue to a topology on X^{\log} making the projection $\tau : X^{\log} \rightarrow X$ given by $\tau(x, F) := x$ a proper map.

8.4. Relating Kato–Nakayama spaces to oriented blowups

There is a morphism

$$\begin{aligned} \phi : X^{\log} &\rightarrow \text{Blo}_D X \\ (x, F) &\mapsto (x, f \mapsto F(\bar{f})) \end{aligned}$$

of topological spaces over X which requires a little explanation. Here $f \in S^1 \mathcal{O}_X(-D)|_x$ is in the circle bundle associated to the fiber $\mathcal{O}_X(-D)|_x \cong \mathbb{C}$ and $\bar{f} \in \mathcal{O}_{X,x}(-D)$ is a lifting of f to the stalk (one shows that any such \bar{f} is actually in $\mathcal{M}_{X,x}$ and that $F(\bar{f})$ doesn't depend on this choice of lifting \bar{f}). If we use the identification

$$S^1 \mathcal{O}_X(D) = \text{Hom}_{S^1}(S^1 \mathcal{O}_X(-D), S^1),$$

then we can think of ϕ as a map from X^{\log} to $S^1 \mathcal{O}_X(D)$; one then shows that this ϕ is continuous and that ϕ factors through $\text{Blo}_D X \subseteq S^1 \mathcal{O}_X(D)$.

has a natural

When X is a smooth analytic space and D is a smooth divisor, the map ϕ is easily seen to be an isomorphism since one can reduce to the case $(X, D) = (\mathbb{C}, 0)$ on formal grounds. Slightly more generally, if X is smooth, but D is only a normal crossings divisor, then locally we can write D as a union of smooth divisors D_1, \dots, D_i which look like the first i coordinate hyperplanes in \mathbb{C}^n ($n = \dim X$), and we can define a variant of the oriented real blowup

$$\mathrm{Blo}'_D X := (\mathrm{Blo}_{D_1} X) \times_X \cdots \times_X (\mathrm{Blo}_{D_i} X)$$

and a map $\phi : X^{\mathrm{log}} \rightarrow \mathrm{Blo}'_D X$. In this local picture, the log structure \mathcal{M}_X is the direct sum (in the category of log structures) of the log structures \mathcal{M}_X^j from D_1, \dots, D_i , the associated Kato–Nakayama space X^{log} is the fibered product over X of the $(X, \mathcal{M}_X^j)^{\mathrm{log}}$, and ϕ is just the fibered product over X of the previously constructed isomorphisms $\phi_j : (X, \mathcal{M}_X^j)^{\mathrm{log}} \rightarrow \mathrm{Blo}_{D_j} X$. The locally defined variants can be glued to define a global variant $\mathrm{Blo}'_D X$ of the oriented real blowup and an isomorphism $\phi : X^{\mathrm{log}} \cong \mathrm{Blo}'_D X$ of topological spaces over X .

8.5. Topology, cohomology, and the Kato–Nakayama space

Locally, if $X = \mathbb{C}^n$ and D is the union of the first i coordinate hyperplanes, then D is the zero locus of $(z_1, \dots, z_n) \mapsto z_1 \cdots z_i \in \mathbb{C}$ and we have

$$\begin{aligned} \mathrm{Blo}_D X &= \{(z_1, \dots, z_n, Z) \in \mathbb{C}^n \times S^1 : |z_1 \cdots z_i|Z = z_1 \cdots z_i\} \\ \mathrm{Blo}'_D X &= \{(\bar{z}, \bar{Z}) \in \mathbb{C}^n \times (S^1)^i : |z_j|Z_j = z_j \text{ for } j = 1, \dots, i\}. \end{aligned}$$

In the general normal crossings divisor situation, the fiber of $\tau : X^{\mathrm{log}} \rightarrow X$ over a point $x \in X$ is naturally identified with

$$S^1 N_{D_1/X}|_x \times \cdots \times S^1 N_{D_i/X}|_x,$$

where D_1, \dots, D_i are the branches of D containing x . When $\dim X = n$, a point $y \in \tau^{-1}(x)$ has a neighborhood diffeomorphic to a neighborhood of the origin in $\mathbb{R}_{\geq 0}^i \times \mathbb{R}^{2n-i}$. (Note $i \leq n$, so the depth of the corners in a Kato–Nakayama space is somewhat constrained.) Recall that the *topology* near the origin only depends on whether $i > 0$, but the differentiable structure depends on the actual value of i . In particular, the topological boundary of the manifold X^{log} is given by $\tau^{-1}(D)$, and this manifold with boundary is homotopy equivalent to its interior, so $H^*(X^{\mathrm{log}}) = H^*(X \setminus D)$.

The topology of morphisms of Kato–Nakayama spaces is also very nice, as shown by the following beautiful general result, see [39, Theorem 0.3]:

Theorem 8.2. *Let $f : X \rightarrow Y$ be a log smooth and integral morphism of fine log analytic spaces. Then the associated map $X^{\mathrm{log}} \rightarrow Y^{\mathrm{log}}$ is a topological submersion.*

In fact, Nakayama and Ogus prove a more general result, replacing integrality by K. Kato's notion of *exact morphisms*, see [24, Definition 4.6].

The fact that the topology is nice suggests that one expects nice cohomological implications. This is indeed the original motivation leading Kato and Nakayama to define X^{\log} , see [26, Theorem 0.2 (1)]:

Theorem 8.3. *Let (X, M) be a fine and saturated log scheme with X of finite type over \mathbb{C} . Let F be a constructible sheaf on the log-étale site $X_{\log\text{-ét}}$, and let F^{\log} be its pullback to the topological space X^{\log} . Then for all $q \in \mathbb{Z}$ we have*

$$H^q(X_{\log\text{-ét}}, F) = H^q(X^{\log}, F^{\log}).$$

Very strong results hold true for de Rham cohomology. In fact the Kato–Nakayama space is a model for the log de Rham cohomology of X in the sense that

$$H^*(X^{\log}, \mathbb{C}) = \mathbb{H}^*(X, \wedge^\bullet \Omega_{(X, M)})$$

under mild assumptions on X . We discuss this in Section 9 below.

8.6. Kato–Nakayama spaces of expanded pairs

Given a pair (X, D) consisting of a smooth variety X over \mathbb{C} with a smooth divisor $D \subseteq X$, the notion of an *expanded pair* $t : \mathcal{X} \rightarrow B$ over a base B arises in various relative curve counting theories. The fiber of t over a point $b \in B$ always looks like

$$X[n]_0 = X \prod_D \Delta_1 \prod_D \cdots \prod_D \Delta_n$$

(for an appropriate n), where $\Delta_i = \mathbb{P}(N_{D/X} \oplus \mathcal{O}_D)$ is a \mathbb{P}^1 bundle over X . Both \mathcal{X} and B have natural log structures making t a log smooth map of log schemes. The fiber of $t^{\log} : \mathcal{X}^{\log} \rightarrow B^{\log}$ over a point $c \in \tau_B^{-1}(b)$ looks like

$$X^{\log} \prod_{c_1: S^1 N_{D/X} \cong S^1 N_{D/\Delta_1}} \Delta_1^{\log} \cdots \prod_{c_n: S^1 N_{D/\Delta_{n-1}} \cong S^1 N_{D/\Delta_n}} \Delta_n^{\log},$$

where the choice of $c \in \tau^{-1}(b) \cong (S^1)^n$ determines the choice of orientation reversing S^1 bundle isomorphisms c_1, \dots, c_n . Here each Δ_j has the log structure from the two copies of D , and Δ_j^{\log} is a cylinder bundle over D (better: an I -bundle over D^{\log}).

The action of $(\mathbb{C}^*)^n$ on $X[n]_0$ given by scaling the fibers of the \mathbb{P}^1 bundles Δ_i is an action by isomorphisms of log schemes, so it lifts to an action on Kato–Nakayama spaces. This lifted action is nontrivial on B^{\log} as the $(S^1)^n$ factor of $(\mathbb{C}^*)^n$ acts simply transitively on $\tau_B^{-1}(b) \cong (S^1)^n$. In the usual moduli problems involving expansions, the isotropy group of a point b involves elements of $(\mathbb{C}^*)^n$ such that the induced action on $X[n]_0$ respects a map from a curve to $X[n]_0$, a subscheme of $X[n]_0$, etc., and this isotropy is usually required to be finite to have a good moduli problem. Since $G \cap \mathbb{R}_{>0} = \{\text{Id}\}$ for any finite subgroup G of \mathbb{C}^* , the Kato–Nakayama space of the moduli problem is often representable even

if the moduli problem itself is not. This is always the case for moduli problems involving, say, quotients of sheaves on $X[n]_0$ pulled back from X , since these quotients themselves have no automorphisms and the only isotropy comes from the subgroup of $(\mathbb{C}^*)^n$ preserving the quotient.

9. Log De Rham and hodge structures

DeRham

9.1. Moduli spaces of polarized Hodge structures.

This section owes much to a lecture by Phillip Griffiths [11].

We assume the reader to know some basic concepts of Hodge theory. First of all, we briefly summarize the classical theory of the moduli spaces of polarized Hodge structures.

subsection1.1

9.1.1. The moduli space $M_h = \Gamma \backslash D_h$. Let n be an integer, and let h be a sequence of non-negative integers $(h^{n,0}, h^{n-1,0}, \dots, h^{0,n})$ satisfying $h^{p,q} = h^{q,p}$, called the *Hodge numbers*. Let $H_{\mathbb{Z}}$ be a free abelian group of rank $\sum h^{p,n-p}$, with a non-degenerate bilinear form $Q : H_{\mathbb{Z}} \otimes H_{\mathbb{Z}} \rightarrow \mathbb{Z}$, which is symmetric (resp. anti-symmetric) if n is even (resp. odd). Let $G_{\mathbb{Z}}$ be the group functor $\text{Aut}(H_{\mathbb{Z}}, Q)$ on commutative rings, sending a ring R to the group of automorphisms on the free R -module $H_R := H_{\mathbb{Z}} \otimes R$ preserving the bilinear form Q . It is an affine group scheme of finite type over \mathbb{Z} (which is clear if we write down the matrix representing the bilinear form Q with respect to some basis of $H_{\mathbb{Z}}$). Let Γ be an arithmetic subgroup of $G_{\mathbb{Z}}(\mathbb{Z})$.

what does
it mean?

The set of Hodge structures of weight n on $H_{\mathbb{R}}$ with prescribed Hodge numbers h , such that Q induces a *polarization* on $H_{\mathbb{R}}$ (i.e. it induces a morphism $H_{\mathbb{R}} \otimes H_{\mathbb{R}} \rightarrow \mathbb{R}(-n)$ of Hodge structures, and the bilinear form $Q_C(u, v) := Q(u, Cv)$, where C is the Weil operator, is symmetric and positive definite), is parameterized by the homogeneous space $D_h = G_{\mathbb{R}}/K$, where K is the stabilizer group of a fixed polarized Hodge structure F_0 on $H_{\mathbb{R}}$. See for instance ([37], §3) for these concepts.

This homogeneous space $D = D_h = G_{\mathbb{R}}/K$ has a complex structure defined as follows. It is clear that $Q : H_{\mathbb{R}} \otimes H_{\mathbb{R}} \rightarrow \mathbb{R}(-n)$ is a morphism of Hodge structures if and only if $Q(F^p, F^{n-p+1}) = 0$ for all p . Let $f^p = \sum_{r \geq p} h^{r, n-r}$, and let D^{\vee} , the *compact dual of D* , be the subspace of the product of the Grassmannians $\prod_p \text{Gr}(f^p, H_{\mathbb{R}})$ consisting of all flags F^{\bullet} :

$$\dots \subset F^{p+1} \subset F^p \subset \dots$$

such that $Q(F^p, F^{n-p+1}) = 0$. Then $D^{\vee} \simeq G_{\mathbb{C}}/P$, where P is a parabolic subgroup preserving a fixed flag. This gives D^{\vee} a complex structure. We see that $D \subset D^{\vee}$ is the locus of flags satisfying

- (i) $F^p \cap \overline{F}^{n-p+1} = 0$ (so that $F^p \oplus \overline{F}^{n-p+1} \cong H_{\mathbb{C}}$) for all p , and
- (ii) $Q(\overline{v}, Cv) > 0$ for all $u \neq 0$ in $H_{\mathbb{C}}$.

They are both open conditions, so $D \subset D^\vee$ is an open complex submanifold. The group Γ acts on D_h properly discontinuously, and the quotient $M_h = \Gamma \backslash D_h$ is the moduli space of Γ -equivalence classes of Q -polarized Hodge structures on $H_{\mathbb{C}}$ with Hodge type h . See ([27], 0.3.6, 0.3.7).

9.1.2. Variations of Hodge structure.

Definition 9.1. Let S be a complex manifold. A variation of Hodge structure \mathcal{H} of weight n on S is given by

- a local system $\mathcal{H}_{\mathbb{Z}}$ of free abelian groups of finite rank on S ;
 - a finite decreasing filtration $F^\bullet \mathcal{H}_{\mathcal{O}}$ of the vector bundle $\mathcal{H}_{\mathcal{O}} := \mathcal{H}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_S$ by holomorphic sub-bundles,
- such that the following conditions are satisfied:
- 1) (Griffiths transversality) the natural flat connection $\nabla = d \otimes id_{\mathcal{H}_{\mathbb{Z}}} : \mathcal{H}_{\mathcal{O}} \rightarrow \Omega_S^1 \otimes \mathcal{H}_{\mathcal{O}}$ takes $F^p \mathcal{H}_{\mathcal{O}}$ into $\Omega_S^1 \otimes F^{p-1} \mathcal{H}_{\mathcal{O}}$, for every p ;
 - 2) for each point $s \in S$, the fiber $F^\bullet(s)$ over s is a Hodge structure of weight n .

A polarization of the variation of Hodge structure \mathcal{H} is a locally constant bilinear form

$$\mathcal{Q} : \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{H}_{\mathbb{Z}} \rightarrow \mathbb{Z}$$

such that on each fiber over $s \in S$, it induces a polarization of the fiber Hodge structure.

Suppose we have a polarized family of Hodge structures $(\mathcal{H}, \mathcal{Q} : \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{H}_{\mathbb{Z}} \rightarrow \mathbb{Z})$ of weight n on S , and a global section of the sheaf

$$\Gamma \backslash \underline{\text{Isom}}((\mathcal{H}_{\mathbb{Z}}, \mathcal{Q}), (H_{\mathbb{Z}}, Q)),$$

where $H_{\mathbb{Z}}$ is regarded as a constant sheaf on S , and assume that all the monodromies of this family of Hodge structures on S are contained in Γ . Then there is a period map

$$\varphi : S \rightarrow M_h$$

inducing this family of Hodge structures. This map is locally liftable to D_h .

If $f : X \rightarrow S$ is a projective smooth morphism between quasi-projective complex algebraic manifolds, with a relative hyperplane section $\eta \in H^0(S, R^2 f_* \mathbb{Z})$, then the family of the primitive part $P^n(X_s, \mathbb{Z})$ of the cohomology groups $H^n(X_s, \mathbb{Z})$ modulo torsion form a polarized variation of Hodge structure of weight n on S , and it induces a period map $S \rightarrow M_h$. To be precise, the family of $H^n(X_s, \mathbb{C})$'s are the stalks of $R^n f_*(f^{-1} \mathcal{O}_S)$, and the Hodge filtration on $R^n f_*(f^{-1} \mathcal{O}_S)$ is given by the degenerate spectral sequence

$$E_1^{pq} = R^q f_* \Omega_{X/S}^p \implies R^{p+q} f_*(f^{-1} \mathcal{O}_S),$$

which is induced from the resolution $\Omega_{X/S}^*$ of $f^{-1}\mathcal{O}_S$ (the relative holomorphic Poincaré lemma, see ([8], 3.4)). Since η is a global section, the primitive part form a variation of sub-Hodge structures on S .

9.2. Logarithmic Hodge structures.

One can ask the following question. Let $f : X \rightarrow S$ be a family of projective manifolds, and let S be the complement of a normal crossing divisor D in some compact manifold \bar{S} , and suppose one can extend the family f to a family $\bar{f} : \bar{X} \rightarrow \bar{S}$ which is log smooth (here \bar{S} has the log structure induced by the divisor D). Is it possible to enlarge the moduli space M_h to some \bar{M}_h so that the period map extends to $\bar{\varphi} : \bar{S} \rightarrow \bar{M}_h$?

To study the degenerations of Hodge structures, Kato and Usui introduced the notion of logarithmic Hodge structures.

9.2.1. The ringed space X^{\log} . Let $(X, \alpha : M_X \rightarrow \mathcal{O}_X)$ be an fs log analytic space over \mathbb{C} (for instance the \mathbb{C} -points of an fs log scheme over \mathbb{C}), and let X^{\log} be the set of pairs (x, u) , where $x \in X$ and $u : M_{X,x} \rightarrow S^1$ is a homomorphism of monoids, such that $u(f) = f(x)/|f(x)|$ for $f \in \mathcal{O}_{X,x}^* \subset M_{X,x}$. Here S^1 is the unit circle in the complex plane. Let $\tau : X^{\log} \rightarrow X$ be the function $(x, u) \mapsto x$. For any open $U \subset X$ and $f \in M_X(U)$, there is a function $\arg(f) : \tau^{-1}(U) \rightarrow S^1$ sending $(x, u) \mapsto u(f)$. We give X^{\log} the weakest topology such that the functions τ and $\arg(f)$ are continuous. Over the open set $X^* \subset X$ where the log structure is trivial, the map τ is a homeomorphism, and the section $j^{\log} : X^* \hookrightarrow X^{\log}$ is a homotopy equivalence. The map τ is proper, with fibers $\tau^{-1}(x)$ compact tori $(S^1)^m$, where m is the rank of $\bar{M}_{X,x}^{\text{gp}}$.

*refer back to
rounding
section*

One can define a sheaf of rings $\mathcal{O}_{X^{\log}}$ on X^{\log} . Roughly speaking, this is the subsheaf of rings of $j_*^{\log} \mathcal{O}_{X^*}$ on X^{\log} generated over $\tau^{-1}\mathcal{O}_X$ by “ $\log(q)$ ”, for all $q \in M_X^{\text{gp}}$. See ([27], 2.2.4) for the precise definition.

For example, if $x \in X$ and $y \in \tau^{-1}(x)$, and the free abelian group $\bar{M}_{X,x}^{\text{gp}}$ has rank m and is generated by $f_1, \dots, f_m \in M_{X,x}^{\text{gp}}$, then the stalk $\mathcal{O}_{X^{\log},y}$ is isomorphic to the polynomial ring $\mathcal{O}_{X,x}[\log(f_1), \dots, \log(f_m)]$. This shows that in general, $(X^{\log}, \mathcal{O}_{X^{\log}})$ is not a locally ringed space.

Let Ω_X^1 be the sheaf of log differential forms on the fs log analytic space X , i.e.

$$\Omega_X^1 = (\Omega_X^1 \oplus (\mathcal{O}_X \otimes_{\mathbb{Z}} M_X^{\text{gp}})) / \{(-d\alpha(f), \alpha(f) \otimes f) \mid f \in M_X\}.$$

For a morphism $f : X \rightarrow Y$ of fs log analytic spaces, define

$$\Omega_{X/Y}^1 = \text{Coker}(f^*\Omega_Y^1 \rightarrow \Omega_X^1).$$

They are both coherent \mathcal{O}_X -modules. Let $\Omega_{X/Y}^r$ be the r -th exterior power of $\Omega_{X/Y}^1$, and let

$$\Omega_{X^{\log}/Y^{\log}}^r = \tau^*\Omega_{X/Y}^r = \tau^{-1}\Omega_{X/Y}^r \otimes_{\tau^{-1}\mathcal{O}_X} \mathcal{O}_{X^{\log}}.$$

One can define differential maps and have the log de Rham complex $(\Omega_{X/Y}^\bullet, d)$ (resp. $(\Omega_{X^{\log}/Y^{\log}}^\bullet, d)$) on X (resp. X^{\log}).

For $y \in X^{\log}$ and $x = \tau(y) \in X$, let $\text{sp}(y)$ be the set of all ring homomorphisms $s : \mathcal{O}_{X^{\log}, y} \rightarrow \mathbb{C}$ that extend the evaluation map $\text{ev}_x : \mathcal{O}_{X, x} \rightarrow \mathbb{C}$. Since $\mathcal{O}_{X^{\log}, y}$ is isomorphic to the polynomial ring over $\mathcal{O}_{X, x}$ generated by log of a basis for $\overline{M}_{X, x}$, if we fix an $s_0 \in \text{sp}(y)$, then we have a bijection:

$$s \mapsto (f \mapsto s(\log(f)) - s_0(\log(f))) : \text{sp}(y) \xrightarrow{\sim} \text{Hom}_{\text{group}}(\overline{M}_{X, x}^{\text{gp}}, \mathbb{C}).$$

9.2.2. Log variations of polarized Hodge structure.

LVPHS

Definition 9.2. Let X be an fs log analytic space. A log variation of polarized Hodge structure of weight n on X is given by

- a local system of free abelian groups of finite rank $\mathcal{H}_{\mathbb{Z}}$ on X^{\log} ,
- a bilinear form $\mathcal{Q} : \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{H}_{\mathbb{Z}} \rightarrow \mathbb{Z}$,
- a finite decreasing filtration $F^\bullet \mathcal{H}_{\mathcal{O}}$ of $\mathcal{H}_{\mathcal{O}} := \mathcal{H}_{\mathbb{Z}} \otimes \mathcal{O}_{X^{\log}}$ by $\mathcal{O}_{X^{\log}}$ -submodules,

such that the following conditions are satisfied:

- 1) there exist a locally free \mathcal{O}_X -module \mathcal{E} and a finite decreasing filtration $F^\bullet \mathcal{E}$ by \mathcal{O}_X -submodules, such that $\text{Gr}_p(\mathcal{E})$ is locally free for each p , and

$$F^p \mathcal{H}_{\mathcal{O}} = \tau^* F^p \mathcal{E} = \tau^{-1} F^p \mathcal{E} \otimes_{\tau^{-1} \mathcal{O}_X} \mathcal{O}_{X^{\log}};$$

- 2) for $y \in X^{\log}$ and $x = \tau(y) \in X$, let $s \in \text{sp}(y)$ and let $f_1, \dots, f_r \in M_{X, x} - \mathcal{O}_{X, x}^*$ generate the monoid $\overline{M}_{X, x}$. If the $|\exp(s(\log(f_i)))|$ are sufficient small for all i , then $(\mathcal{H}_{\mathbb{Z}, y}, \mathcal{Q}, F^\bullet(s))$ is a polarized Hodge structure of weight n ;

- 3) the connection $d \otimes \text{id} : \mathcal{H}_{\mathcal{O}} \rightarrow \Omega_{X^{\log}}^1 \otimes_{\mathcal{O}_{X^{\log}}} \mathcal{H}_{\mathcal{O}}$ takes $F^p \mathcal{H}_{\mathcal{O}}$ into $\Omega_{X^{\log}}^1 \otimes_{\mathcal{O}_{X^{\log}}} F^{p-1} \mathcal{H}_{\mathcal{O}}$.

Here $F^\bullet(s)$, the specialization of F at s , is the decreasing filtration of $\mathcal{H}_{\mathbb{C}, y} := \mathbb{C} \otimes_{\mathbb{Z}} \mathcal{H}_{\mathbb{Z}, y}$ defined by $F^p(s) = \mathbb{C} \otimes_{\mathcal{O}_{X^{\log}, y}} F^p \mathcal{H}_{\mathcal{O}}$. For a fixed point $y \in X^{\log}$, the family $(\mathcal{H}_{\mathbb{Z}, y}, \mathcal{Q}, F^\bullet(s))_{s \in \text{sp}(y)}$ is called a polarized log Hodge structure on the log point $(x, M_{X, x})$; this is the same as a log variation of polarized Hodge structure on the log point $(x, M_{X, x})$.

Log variations of polarized Hodge structure arise from geometry in the following way. Let $f : X \rightarrow Y$ be a projective log smooth morphism between fs log analytic spaces, and we fix a line bundle on X which is relatively ample over Y . By a theorem of Kajiwara and Nakayama, for every integer n , the sheaf $R^n f_*^{\log} \mathbb{Z}$ is a local system on Y^{\log} . We take $\mathcal{H}_{\mathbb{Z}}$ to be $R^n f_*^{\log} \mathbb{Z}$ modulo torsion, take \mathcal{Q} to be the pairing induced by the fixed ample line bundle, take \mathcal{E} to be $R^n f_*(\Omega_{X/Y}^\bullet)$, with filtration $F^p \mathcal{E} = R^n f_*(\Omega_{X/Y}^{\geq p}) \subset \mathcal{E}$, and take $F^p \mathcal{H}_{\mathcal{O}}$ to be $\tau^* F^p \mathcal{E}$. Then by a theorem of Kato, Matsubara and Nakayama, this is a log variation of polarized Hodge structure on Y .

Is it really this general log smoothness is enough? Double check assumptions

9.3. Kato-Usui spaces.

We fix $n, h, H_{\mathbb{Z}}, Q, G_{\mathbb{Z}}, D$ and D^{\vee} as in (9.1.1). Let $\mathfrak{g}_{\mathbb{R}} = \text{Lie}(G_{\mathbb{R}})$. A subset $\sigma \subset \mathfrak{g}_{\mathbb{R}}$ is called a *nilpotent cone* if it is a cone

$$\sigma = \sum_{i=1}^n \mathbb{R}_{\geq 0} N_i$$

generated by mutually commutative nilpotent operators $N_i \in \mathfrak{g}_{\mathbb{R}} \subset \text{End}(H_{\mathbb{R}})$. Let Γ be a *neat subgroup* of $G_{\mathbb{Z}}(\mathbb{Z})$, i.e. for every element $\gamma \in \Gamma$, its eigenvalues on $H_{\mathbb{C}}$ generate a torsion-free subgroup of \mathbb{C}^* .

9.3.1. Nilpotent orbits.

nilp-orbit

Definition 9.3. Let $\sigma = \sum_i \mathbb{R}_{\geq 0} N_i$ be a nilpotent cone. A subset $Z \subset D^{\vee}$ is called a σ -*nilpotent orbit*, if there exists an $F_0 \in D^{\vee}$ such that

- $Z = \exp(\sum_i \mathbb{C} N_i) F_0$,
- $N F_0^p \subset F_0^{p-1}$ for all $p \in \mathbb{Z}$ and $N \in \sigma$,
- $\exp(\sum_i z_i N_i) F_0 \in D$ if $\text{Im}(z_i) \gg 0$ for all i .

We also call the pair (σ, Z) a *nilpotent orbit*.

Let Σ be a *fan* in $\mathfrak{g}_{\mathbb{Q}}$, i.e. $\Sigma \neq \emptyset$ is a set of rational nilpotent cones in $\mathfrak{g}_{\mathbb{R}}$ (namely, those generated by nilpotent operators in $\mathfrak{g}_{\mathbb{Q}}$) such that

- if $\sigma \in \Sigma$, then all faces of σ are in Σ ,
- for $\sigma, \sigma' \in \Sigma$, the intersection $\sigma \cap \sigma'$ is a face of both σ and σ' ,
- for every $\sigma \in \Sigma$, we have $\sigma \cap (-\sigma) = 0$.

One can then define the set $D_{h, \Sigma}$ (or just D_{Σ} , if there is no confusion) of *nilpotent orbits in the directions in Σ* to be the set of nilpotent orbits (σ, Z) where $\sigma \in \Sigma$. There is a natural injection

$$F \mapsto (0, \{F\}) : D \hookrightarrow D_{\Sigma}.$$

9.3.2. The moduli space M_{Σ} . Let Σ be a fan in $\mathfrak{g}_{\mathbb{Q}}$ and let $\Gamma \subset G_{\mathbb{Z}}(\mathbb{Z})$ be a subgroup. Then we say that Γ is *compatible with Σ* if for every $\gamma \in \Gamma$ and $\sigma \in \Sigma$, we have $\text{Ad}(\gamma)(\sigma) \in \Sigma$. In this case, there is an action of Γ on D_{Σ} given by

$$(\sigma, Z) \xrightarrow{\gamma} (\text{Ad}(\gamma)(\sigma), \gamma Z).$$

We say that Γ is *strongly compatible with Σ* if every cone $\sigma \in \Sigma$ is generated by elements in $\log \Gamma$. Kato and Usui showed that when Γ is strongly compatible with Σ and the arithmetic subgroup Γ is neat, the quotient set $\Gamma \backslash D_{\Sigma}$ can be given the structure of a log locally ringed space over \mathbb{C} , in fact a *log manifold* (see ([27]; 3.5.7)). Roughly speaking, a log manifold is a log locally ringed space over \mathbb{C} , which is locally isomorphic to the “zero locus” of some log differential forms on a log smooth analytic space.

Informally speaking, Kato and Usui proved the following. First, there is a one-to-one correspondence between D_Σ and the set of polarized log Hodge structures of the given type. Second, if $\overline{X} \rightarrow \overline{S}$ is a log smooth family extending the projective smooth family $X \rightarrow S$, where $S \subset \overline{S}$ is the complement of a normal crossing divisor, then the period map extends to $\overline{S} \rightarrow M_\Sigma$. We briefly explain the first part in the following.

We shall show how to get a nilpotent orbit from a polarized log Hodge structure on a log point ([27], 0.4.24). Let x be an fs log point with log structure M_x . Then \overline{M}_x is a sharp fs monoid and $\overline{M}_x^{\text{gp}}$ a free abelian group of finite rank, say r . Fix $y \in x^{\text{log}}$. We have $x^{\text{log}} = \text{Hom}(\overline{M}_x^{\text{gp}}, S^1) \simeq (S^1)^r$ and hence $\pi_1(x^{\text{log}}) = \text{Hom}(\overline{M}_x^{\text{gp}}, \mathbb{Z}) \simeq \mathbb{Z}^r$. Let $\pi_1^+(x^{\text{log}}) \subset \pi_1(x^{\text{log}})$ be the subset consisting of those homomorphisms $a : \overline{M}_x^{\text{gp}} \rightarrow \mathbb{Z}$ that take \overline{M}_x into \mathbb{N} ; this subset is an fs monoid.

Let $(H_{\mathbb{Z}}, Q, F^\bullet H_\theta)$ be a polarized log Hodge structure on x . Let $(h_i)_{i=1}^n$ be a family of generators for $\pi_1^+(x^{\text{log}})$ and $s_0 \in \text{sp}(y)$. Let z_1, \dots, z_r be complex numbers, and let $s \in \text{sp}(y)$ be such that

$$s \left(\frac{\log(f)}{2\pi i} \right) - s_0 \left(\frac{\log(f)}{2\pi i} \right) = \sum_{i=1}^r z_i h_i(f), \quad \text{for } f \in \overline{M}_x^{\text{gp}}.$$

Let $N_i : H_{\mathbb{Q}, y} \rightarrow H_{\mathbb{Q}, y}$ be the logarithm of h_i . Then we have

$$F(s) = \exp \left(\sum_{i=1}^n z_i N_i \right) F(s_0),$$

which shows that $(F(s))_{s \in \text{sp}(y)}$ is an orbit of filtrations under $\exp(\sigma \otimes \mathbb{C})$ for $\sigma = \sum_i \mathbb{R}_{\geq 0} N_i$. Moreover, the condition 2) in (9.2) implies that $F(s) \in D$ if $\text{Im}(z_i) \gg 0$ for all i , and the condition 3) in (9.2) implies that $NF(s_0)^p \subset F(s_0)^{p-1}$ for all $p \in \mathbb{Z}$ and $N \in \sigma$. In other words, the family $(F(s))_s$ is a σ -nilpotent orbit.

10. The main component of moduli spaces

MainComp

10.1. Moduli: compactness and main components

In Section 4, we gave an overview of F. Kato's work [23] in which he uses log geometry to compactify the moduli space $\mathcal{M}_{g,n}$ of curves. Specifically, he shows that the moduli space of log smooth curves agrees with the Deligne-Mumford compactification $\overline{\mathcal{M}}_{g,n}$. The key philosophic idea in that section was that since moduli spaces of log smooth objects already includes degenerate objects, it is reasonable to expect that such a moduli space is a compactification of the moduli of objects with trivial log structure.

While the Deligne-Mumford space of stable curves $\overline{\mathcal{M}}_{g,n}$ turns out to be irreducible, it is an unfortunate fact of life that if \mathfrak{X} is a moduli space of higher dimensional objects, then moduli-theoretic "compactifications" $\overline{\mathfrak{X}}$ of \mathfrak{X} tend to have many irreducible components. If \mathfrak{X} is irreducible, then it sits entirely within

one of these many components of $\tilde{\mathfrak{X}}$ and so it is natural then to ask if this “main component” can itself be given a moduli interpretation.

In Section 4.5 we stated a second philosophic principle: log geometry controls degenerations; that is, moduli of log smooth objects does not incorporate “too many” degenerate objects. This provides a type of converse to the aforementioned philosophy that moduli of log smooth objects should be compact. As explained in Section 4.5, the log structure gives us a fighting chance to show that our moduli space is irreducible (although it is of course too naïve to expect that moduli of log smooth objects is always irreducible). Combining the two principles, one may hope that if \mathfrak{X} is an irreducible moduli space and $\tilde{\mathfrak{X}}$ a moduli-theoretic compactification, then by appropriately incorporating log structures into the objects parameterized by $\tilde{\mathfrak{X}}$, one will isolate the main component.

This technique of using log geometry to isolate the main component of a moduli space has been carried out by M. Olsson in several different settings. In [41], Olsson gives a moduli interpretation to the normalization of the main component of the toric Hilbert scheme. In [48], he isolates the normalization of the main component of V. Alexeev’s compactification of the moduli space of principally polarized abelian varieties given in [3]; he further constructs a moduli-theoretic irreducible compactification of the moduli space of abelian varieties with higher degree polarization. In [44], he gives an irreducible modular compactification of the moduli space of polarized K3 surfaces.

10.2. Example: the toric Hilbert scheme

Our goal in this section is to explain the technique of isolating the main component of a moduli space by following Olsson’s work [41]. We begin with the definition of the toric Hilbert scheme. Let k be a field and let P and Q be finitely-generated integral monoids with Q sharp and P^{gp} and Q^{gp} torsion-free. Fix a surjective morphism $\pi : P \rightarrow Q$. This yields a closed immersion from $A_Q := \text{Spec } k[Q]$ to $A_P := \text{Spec } k[P]$, which is T_Q -equivariant, where T_Q (resp. T_P) denotes the torus associated to Q^{gp} (resp. P^{gp}). Consider the functor \mathcal{H} whose S -valued points are diagrams

$$\begin{array}{ccc} Z & \xrightarrow{i} & A_{P,S} \\ & \searrow g & \downarrow \\ & & S \end{array}$$

where i is a T_Q -invariant closed immersion and for every $q \in Q^{gp}$, the q -eigenspace of $g_* \mathcal{O}_Z$ is a finitely-presented projective \mathcal{O}_S -module of rank 1 if $q \in Q$ and rank 0 otherwise. By [14, Thm 1.1], this functor is representable by a quasi-projective scheme, which we call the *toric Hilbert scheme*.

Given a closed subscheme Z of $A_{P,S}$ as above, we can move Z by the action of T_P on $A_{P,S}$. This yields an action of T_P on \mathcal{H} . Since Z is T_Q -invariant, this action factors through $T_K = T_P/T_Q$, where K denotes the kernel of π^{gp} . We therefore obtain a map

$$T_K \longrightarrow \mathcal{H}$$

by letting $u \in T_K$ act on the distinguished point of $\mathcal{H}(k)$ given by the closed immersion $A_Q \rightarrow A_P$. By [6, 3.6(2)], this map is an open immersion. Therefore the normalization \mathcal{S} of the scheme-theoretic closure of its image is a normal toric variety, and hence carries a natural fs log structure $\mathcal{M}_{\mathcal{S}}$ which makes it log smooth over $\text{Spec } k$ (endowed with the trivial log structure). The goal of [41] is to give a moduli-theoretic interpretation of $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$.

Consider the functor \mathcal{H}^{log} on the category of fs log schemes over k whose $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$ -valued points are given by diagrams

$$\begin{array}{ccc} (Z, \mathcal{M}_Z) & \xrightarrow{i} & (A_P, \mathcal{M}_{A_P}) \times (\mathcal{S}, \mathcal{M}_{\mathcal{S}}) \\ & \searrow g & \downarrow \\ & & (\mathcal{S}, \mathcal{M}_{\mathcal{S}}) \end{array}$$

where the underlying maps on schemes defines a point of $\mathcal{H}(\mathcal{S})$, where g is log smooth and integral, and where the map

$$P \longrightarrow \mathcal{M}_{(Z, \mathcal{M}_Z)/(\mathcal{S}, \mathcal{M}_{\mathcal{S}})} := \text{coker}(g^* \mathcal{M}_{\mathcal{S}} \rightarrow \mathcal{M}_Z)$$

induced by i factors through Q . The main theorem of [41] is then

thm:mainthm

Theorem 10.1 ([41, Thm 1.6]). *The functor \mathcal{H}^{log} is representable by $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$.*

The first step in proving this theorem is to obtain a map from the functor points $\mathcal{H}_{\mathcal{S}}$ of $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$ to \mathcal{H}^{log} . This is done by showing that the pullback to \mathcal{S} of the universal family over \mathcal{H} yields a point of $\mathcal{H}^{log}(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$. Explicitly, if $i : \mathcal{Z} \rightarrow A_P \times \mathcal{S}$ is the pullback of the universal family, Olsson constructs a log structure on \mathcal{Z} as follows. Since \mathcal{S} is a toric variety with torus T_K , it can be covered by open affines of the form $\text{Spec } k[L]$ with L a submonoid of K whose associated group is K . Olsson proves ([41, 2.4]) that over such an open affine, the closed immersion i is given by

$$\mathcal{Z} \times_{\mathcal{S}} \text{Spec } k[L] = \text{Spec } k[E_L] \longrightarrow \text{Spec } k[P \oplus L] = A_P \times \text{Spec } k[L],$$

where E_L is the image of $P \oplus L$ in P^{gp} under the map $(p, \ell) \mapsto p + \ell$.

Example 10.2. Let $\pi : \mathbb{N}^2 \rightarrow \mathbb{N}$ send e_1 to 2 and e_2 to 1. Then the kernel K of π^{gp} is all integer multiples of $2e_2 - e_1$. Let L be the submonoid of K consisting of the non-negative multiples of $e_1 - 2e_2$ and let M be the submonoid consisting of the non-positive multiples. Then E_L is generated by e_1, e_2 , and $2e_2 - e_1$; hence,

$$\text{Spec } k[E_L] \simeq k[x, y, z]/(xy - z^2).$$

We see that E_M is freely generated by e_2 and $e_1 - 2e_2$, so $\text{Spec } k[E_M] \simeq \mathbb{A}_k^2$.

We see then that $\mathcal{Z} \times_S \text{Spec } k[L]$ carries a natural log structure. These log structures glue to give a log structure on \mathcal{Z} (by [41, Lemma 2.8]) which makes i a closed immersion of log schemes. We therefore obtain a morphism of functors

$$F : \mathcal{H}_S \longrightarrow \mathcal{H}^{log}.$$

The next step in proving Theorem 10.1 above is to work out the deformation theory of \mathcal{H}^{log} . Olsson shows ([41, §4]) that if $(\text{Spec } B_0, \mathcal{M}_{B_0}) \rightarrow (\text{Spec } B, \mathcal{M}_B)$ is a strict closed immersion defined by a square zero ideal \mathcal{I} , then every object of $\mathcal{H}^{log}(\text{Spec } B_0, \mathcal{M}_{B_0})$ lifts to an object of $\mathcal{H}^{log}(\text{Spec } B, \mathcal{M}_B)$; moreover, the set of lifts forms a torsor under $\text{Hom}(K, \mathcal{I})$. Since S is a toric variety with torus T_K , we see

$$\Omega_{(S, \mathcal{M}_S)/k}^1 = \mathcal{O}_S \otimes_{\mathbb{Z}} K.$$

As a result, every object of $\mathcal{H}_S(\text{Spec } B_0, \mathcal{M}_{B_0})$ also lifts to an object of $\mathcal{H}_S(\text{Spec } B, \mathcal{M}_B)$ and the set of lifts forms a torsor under $\text{Hom}(K, \mathcal{I})$. This shows

thm:loget

Theorem 10.3 ([41, 4.4]). *The functor F is formally log étale.*

Olsson then makes use of Theorem 10.3 to show that F is injective and surjective, thereby proving Theorem 10.1. We explain his proof of injectivity as it gives a concrete instance of the philosophy and diagram of Section 4.5. Suppose f and g are two morphisms from an fs log scheme (T, \mathcal{M}_T) to (S, \mathcal{M}_S) such that $Ff = Fg$. We show that $f = g$. Since S is locally of finite presentation over k , a limit argument allows us to assume T is of finite type over k . Since \mathcal{H}^{log} is limit preserving (by [41, §6]), another limit argument shows that it is enough to check $f = g$ at the completion of T at every point. Since T is assumed to be of finite type over k (hence Noetherian), we have reduced to the case $T = \text{Spec } R$ for R a complete local Noetherian ring. To check $f = g$, it now suffices to show that this is true after precomposing with all maps $\text{Spec } R/\mathfrak{m}^n \rightarrow \text{Spec } R$, where \mathfrak{m} is the maximal ideal of R . Hence, we may assume $T = \text{Spec } B$, where B is a local Artin ring. If $i : (\text{Spec } B_0, \mathcal{M}_{B_0}) \rightarrow (\text{Spec } B, \mathcal{M}_B)$ is a strict closed immersion with square zero ideal, and if $fi = gi$, then the diagram

$$\begin{array}{ccc} (\text{Spec } B_0, \mathcal{M}_{B_0}) & \xrightarrow{fi=gi} & (S, \mathcal{M}_S) \\ i \downarrow & \nearrow f & \downarrow F \\ (\text{Spec } B, \mathcal{M}_B) & \xrightarrow{g} & \mathcal{H}^{log} \end{array}$$

commutes. By Theorem 10.3, we see then that $f = g$. We can therefore assume that $T = \text{Spec } k'$ with k' a field. It further suffices to assume that k' is separably closed, in which case there is an fs sharp monoid L and a morphism $L \rightarrow \mathcal{M}_T$ inducing an isomorphism of L with $\bar{\mathcal{M}}_T$.

Consider the complete local ring $R' = k'[[L]]$ defined as the completion of $k'[L]$ at the maximal ideal \mathfrak{m}' which is the kernel of the morphism $k'[L] \rightarrow k'$ sending all non-zero elements of L to 0. The scheme $\text{Spec } R'$ carries a natural log structure $\mathcal{M}_{R'}$ defined as the pullback of the canonical log structure on $\text{Spec } \mathbb{Z}[L]$. By Theorem 10.3 and log smoothness of $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$, we obtain morphisms \tilde{f} and \tilde{g} from $(\text{Spec } R', \mathcal{M}_{R'})$ to $(\mathcal{S}, \mathcal{M}_{\mathcal{S}})$ such that $F\tilde{f}_n = F\tilde{g}_n$ for every n , where \tilde{f}_n (resp. \tilde{g}_n) denotes the reduction of \tilde{f} (resp. \tilde{g}) modulo \mathfrak{m}'^n .

In other words, we have replaced our original T , f , and g by $\text{Spec } R'$, \tilde{f} , and \tilde{g} . So, we are once again in the case where T is the spectrum of a complete local Noetherian ring. What have we gained, then, over the previous case when $T = \text{Spec } R$ above? We have succeeded in replacing an arbitrary complete local Noetherian ring R by a very special one, namely $k'[[L]]$. We saw these special rings come up in the diagram of Section 4.5.

Let $h : \mathcal{S} \rightarrow \mathcal{H}$ be the natural morphism (recall that \mathcal{S} is the normalization of the main component of \mathcal{H}). Since $F\tilde{f}_n = F\tilde{g}_n$ for all n , the very definition of \mathcal{H}^{log} shows that $f\tilde{f}_n = h\tilde{g}_n$ for all n . Therefore, $h\tilde{f} = h\tilde{g}$. Since $k'[[L]]$ is normal and \mathcal{S} is separated, to show that $\tilde{f} = \tilde{g}$, it suffices to show they are equal on a dense open subset. Since the generic point of $k'[[L]]$ has trivial log structure, it factors through the torus T_K of \mathcal{S} . The map h , however, is an open immersion over T_K , and so \tilde{f} and \tilde{g} agree on the generic point of $k'[[L]]$.

11. Twisted curves and log twisted curves

Roots

Twisted curves are a central object in the theory of *twisted stable maps* [2, 5, 1]: in order to have a complete moduli space of stable maps $C \rightarrow X$ of type Γ , where X is a proper tame stack with projective coarse moduli space and $\Gamma = (g, n, \beta)$ are the relevant discrete data, one must allow the curve C itself to be a certain type of stack, called *twisted curve*.

The original treatments of twisted curves relied on ad-hoc methods. The more recent approach of [1] relies on a method introduced in [47], which uses a construction with logarithmic structures.

11.1. Twisted curves

For simplicity we will stick with the case of Deligne–Mumford stacks.

First consider the geometric objects: fix an algebraically closed field k .

:twisted-curve

Definition 11.1. A *twisted curve* over k is a tame, purely 1-dimensional Deligne–Mumford stack \mathcal{C}/k , with at most nodes as singularities, satisfying the following conditions:

- (1) Let $\pi : \mathcal{C} \rightarrow C$ be the morphism to the coarse moduli space. Then $\mathcal{C}^{\text{sm}} = \pi^{-1}C^{\text{sm}}$, and $\pi : \mathcal{C} \rightarrow C$ is an isomorphism over a dense open subset of C .

- (2) Consider a node $\bar{x} \rightarrow C$, where the strictly henselian local ring $\mathcal{O}_{C, \bar{x}}$ is the strict henselization of $k[x, y]/(xy)$. Then

$$\mathcal{C} \times_C \text{Spec } \mathcal{O}_{C, \bar{x}} \simeq \left[\text{Spec } \mathcal{O}_{C, \bar{x}}[z, w]/(zw, z^m - x, w^m - y) \right] / \mu_m,$$

where $\zeta \in \mu_m$ acts by $(z, w) \mapsto (\zeta z, \zeta^{-1}w)$.

An action such as (2) above is called *balanced* - it is crucial to our discussion of log structures below. Note that \mathcal{C} may have a stack structure at isolated smooth points as well - such points will behave like $[\mathbb{A}^1/\mu_a]$, where μ_a acts by multiplication.

Over a general base S twisted curves are detected by their geometric fibers: a *twisted curve* $\mathcal{C} \rightarrow S$ is a flat, tame Deligne–Mumford stack locally of finite presentation, all of whose geometric fibers are twisted curves as in the definition above.

The *genus* of \mathcal{C} is simply the genus of C . One typically needs to consider n -pointed twisted curves, where the markings are described in families as follows:

Definition 11.2. An n -pointed twisted curve \mathcal{C}/S marked by disjoint closed substacks $\{\Sigma_i\}_{i=1}^n$ in \mathcal{C} is assumed to satisfy the following:

- (1) the Σ_i are contained in the smooth locus \mathcal{C}^{sm} ,
- (2) each Σ_i is a tame étale gerbe over S , and
- (3) $\mathcal{C}_{gen} := \mathcal{C}^{sm} \setminus \cup_i \Sigma_i \rightarrow C$ is an open embedding.

Remark 11.3. When $S = \text{Spec } k$ where $k = \bar{k}$, then $\Sigma_i = B\mu_{a_i}$, and moreover a_i is locally constant in families.

Remark 11.4. When $(\mathcal{C}/S, \{\Sigma_i\})$ is an n -pointed twisted curve, then the coarse moduli space of Σ_i is isomorphic to S . This means that the composite morphism $\Sigma_i \rightarrow \mathcal{C} \rightarrow C$ factors through a section $p_i : S \rightarrow C$. It follows that $(C, \{p_i\})$ is an n -pointed curve in the usual sense. This gives a functor

$$(\mathcal{C}/S, \{\Sigma_i\}) \mapsto (C, \{p_i\})$$

One can ask oneself: what does one need in order to recover a twisted n -pointed curve $(\mathcal{C}/S, \{\Sigma_i\})$ from a usual n -pointed curve $(C, \{p_i\})$? In other words, can we enrich the functor above to something like

$$(\mathcal{C}/S, \{\Sigma_i\}) \mapsto (C, \{p_i\}) + ?$$

which is nice and explicit and actually an equivalence of categories?

The stack structure at the marking definitely needs the data of the integers a_i , but in fact this is all that is necessary for the markings: near p_i , the curve \mathcal{C} is canonically isomorphic to the root stack $\mathcal{C}(\sqrt[a_i]{p_i})$. If x is a local generator of the ideal of p_i , then Zariski locally we have

$$\mathcal{C} \simeq \left[\text{Spec } \mathcal{O}_C[z]/(z^{a_i} - x) \right] / \mu_{a_i}.$$