The Category of R-modules has enough injectives.

(Following notes of Arthur Ogus).

Def: An object I in an abelian category C is *injective* if the functor
$$h_I : C \to Ab$$
by $\quad h_I(M) = \mathrm{Hom}(M, I) \quad$ is exact.

Def: Given an object $M \in Ob(C)$, an *injective resolution* is an exact sequence
$$0 \to M \to I_0 \to I_1 \to I_2 \to \cdots$$
where I_0 is injective $\forall$ 0. We say the category C *has enough injectives* if all objects have injective resolutions

Main theorem: Let R be a ring. Then the category of
(✡) $\quad R$-modules has enough injectives.

To prove this, we prove
(1) True for $R = \mathbb{Z}$, in such a way that implies $\mathbb{Q}/\mathbb{Z}$ is injective + $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \neq 0 \;\; \forall$ nonzero $M \in Ab = \mathbb{Z}$-mod.
(2) If there exists an injective R-module I s.t. $\mathrm{Hom}_R(M, I) \neq 0 \;\; \forall \; M \neq 0$, then (✡) holds.
(3) For any ring R, the R-module
$$I = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$$
satisfies the condition in (2).

Proof of (1).

We work in Ab.

Theorem: An abelian group G is injective $\iff$ it is divisible ($\forall\ n \neq 0$, the map $G \xrightarrow{\times n} G$ is surjective).

Pf: Suppose G is injective, so that

$$0 \to N \to M \quad \text{exact}$$

gives

$$\mathrm{Hom}_{\mathbb{Z}}(M,G) \to \mathrm{Hom}_{\mathbb{Z}}(N,G) \to 0 \quad \text{exact.}$$

So

$$0 \to n\mathbb{Z} \xrightarrow{i} \mathbb{Z} \quad \text{exact}$$

gives

$$\tau \qquad \varphi \longmapsto \varphi|_{n\mathbb{Z}} \qquad \tau|_{n\mathbb{Z}}$$

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z},G) \arrow[r] \arrow[d, "\wr"'] & \mathrm{Hom}_{\mathbb{Z}}(n\mathbb{Z},G) \arrow[r] \arrow[d, "\wr"'] & 0 \\
G \arrow[r, "\times n"] & G \arrow[r] & 0
\end{tikzcd}$$

$$\varphi \mapsto \varphi(1) \qquad \psi \mapsto \psi(n)$$

$g = \tau(1)$ for some τ $\qquad \mapsto g$, $\ \tau(n) = n\tau(1) = ng$

which shows that G is divisible. Conversely, assume G is divisible, and suppose given

$$0 \to N \to M,$$

to show $\mathrm{Hom}(M,G) \to \mathrm{Hom}(N,G)$ is surjective. So suppose

$$\varphi : N \to G$$

is a group hom, where $N \leq M$. Define a poset $(S, \leq)$

as follows: the set consists of pairs

$$(N', \psi),$$

where $N \leq N' \leq M$, $\psi: N' \to G$ with $\psi|_N = \varphi$, and

$$(N', \psi') \preceq (N'', \psi'') \iff N' \leq N'' \text{ and}$$

$\psi''|_{N'} = \psi'$. Now given an ascending chain

$$\cdots \preceq (A_i, \psi_i) \preceq (A_{i+1}, \psi_{i+1}) \preceq \cdots,$$

we form a max'l elt $(A, \overline{\psi})$ by

$$A = \bigcup_i A_i,$$

and $\overline{\psi}(a) = \psi_i(a)$ if $a \in A_i$. So applying Zorn's lemma, the poset $(S, \preceq)$ has a maximal element $(H, \overline{\varphi})$. We claim $H = M$. Otherwise, suppose $x \in M \setminus H$. We can extend $\overline{\varphi}$ to $H + \langle x \rangle$ as follows: if $\langle x \rangle \cap H = e$, then $H + \langle x \rangle = H \oplus \langle x \rangle$ and there is the obvious extension; otherwise pick minimal n s.t. $nx \in H$, then by divisibility of G, $\exists\, g \in G$ s.t. $ng = \overline{\varphi}(nx)$; define $\overline{\varphi}': H + \langle x \rangle \to G$ by $\overline{\varphi}'(x) = g$. This contradicts maximality of H, so $H = M$. ///

Corollary: $\mathbb{Q}/\mathbb{Z}$ is injective.

<u>Lemma</u>: $\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z}) \neq 0$ $\forall$ nonzero abelian groups G.

<u>Pf</u>: Let $g \in G$, then

$$0 \to \langle g \rangle \to G$$

gives

$$\mathrm{Hom}_{\mathbb{Z}}(G, \mathbb{Q}/\mathbb{Z}) \xrightarrow{\text{restriction}} \mathrm{Hom}_{\mathbb{Z}}(\langle g \rangle, \mathbb{Q}/\mathbb{Z}) \to 0 \ ;$$

by injectivity of $\mathbb{Q}/\mathbb{Z}$, so we only need to prove lemma for cyclic groups. Given n, find $\varphi \in \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \mathbb{Q}/\mathbb{Z}$ with $\varphi(1) = \frac{1}{n}$, so $\varphi(n) = 0$,

get

$$\begin{tikzcd} \mathbb{Z} \arrow[d] \arrow[dr, "\varphi"] & \\ \mathbb{Z}/n \arrow[r, "\overline{\varphi}"] & \mathbb{Q}/\mathbb{Z}, \end{tikzcd}$$

$\overline{\varphi} \neq 0$. ///

(2) Suppose $\exists$ an injective R-module I s.t. $h_I(M) := \mathrm{Hom}_R(M, I) \neq 0 \ \forall M \neq 0$, then (*) holds.

<u>Required lemma:</u>

Suppose I is as stated. Then for any M, the canonical map $\eta: M \to h_I h_I(M)$ is injective.

<u>Pf:</u> Let $K = \ker \eta$, so we have a diagram

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, "\eta_K"'] & M \arrow[d, "\eta"] \\ h_I h_I(K) \arrow[r] & h_I h_I(M). \end{tikzcd}$$

Since I injective, $h_I(M) \to h_I(K)$ is <u>surjective</u>, so $h_I h_I(K) \to h_I h_I(M)$ is <u>injective</u>. By commutativity of the diagram, this means

$$K \to h_I h_I(K) \quad \text{is the}$$

0-map. This implies $h_I h_I(K) = 0$: if $\hat{\psi} \in h_I h_I(K)$ satisfies $\hat{\psi}(\varphi) \neq 0$ for some $\varphi : K \to I$, then pick k s.t. $\varphi(k) \neq 0$. But then

$$\eta_K(k)(\varphi) = \varphi(k) \neq 0,$$

so $\eta_K(k) \neq 0$, contradiction, so $h_I h_I(K) = 0$. By hypothesis on I, this means $h_I(K) = 0 \Rightarrow K = 0$. ///

Now we prove (2).

<u>Proof of (2):</u> Let M be a nonzero R-module, we want an injection $M \to J$ for some injective R-module J. Fix I as in the statement. Find a free resolution

$$R^{(T)} \to h_I(M) \to 0$$

so as always,

$$0 \to h_I h_I(M) \to h_I(R^{(T)})$$

while $h_I(R^{(T)}) = \mathrm{Hom}_R(R^{(T)}, I) \cong \prod_T I$ is injective. Therefore

$$M \to h_I h_I(M) \to h_I(R^{(S)}) \quad \text{is the required}$$

injection. ////

injective by green lemma.

<u>Proof of 13.7</u>: To finish, we need to show the existence of an injective R-module I s.t. $h_I(M) \neq 0$ $\forall$ $M \neq 0$.

<u>Claim</u>: $I = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ works.

<u>Pf</u>: For $M \neq 0$,

$$h_I(M) = \mathrm{Hom}_R(M, I) = \mathrm{Hom}_R(M, \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z}))$$

(tensor-hom adjunction) $\cong \mathrm{Hom}_{\mathbb{Z}}(M \otimes_R R, \mathbb{Q}/\mathbb{Z})$

$$= \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z}) \neq 0.$$

Moreover, the exactness of $\mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ implies exactness of $\mathrm{Hom}_R(-, I)$. ///