MATH 251 ALGEBRA I
TAKE-HOME MIDTERM

You may consult your notes, books, homework. Return whatever you have on Monday October 31, 2005 in class or in my mailbox. This is long, so start early.

Numbered exercises are from Dummit and Foote.

(1) p. 53 ex. 14
(2) p. 61 ex. 26
(3) (Goursat’s Lemma, from Lang p. 75 ex. 5) Let $G_1, G_2$ be groups, $H < G_1 \times G_2$ such that the two projections $p_i : H \to G_i$ are surjective. Write $N_1 = \text{Ker}(p_2)$ and $N_2 = \text{Ker}(p_1)$.
   (a) Show that $p_i$ maps $N_i$ isomorphically to a normal subgroup of $G_i$.
   (b) Show that the image of $H$ in $G_1/N_1 \times G_2/N_2$ is the graph of an isomorphism $G_1/N_1 \to G_2/N_2$.

(4) p. 101 ex. 7
(5) p. 110 ex. 10
(6) p. 132 ex. 30
(7) p. 148 ex. 40
(8) (a) Find a composition series for $GL_2(\mathbb{Z}/5\mathbb{Z})$
    (b) Find a composition series for $GL_2(\mathbb{Z}/25\mathbb{Z})$
    (c) Find a composition series for $GL_2(\mathbb{Z}/125\mathbb{Z})$
        (to what extent is 5 important here?)

(9) Let $P$ be a $p$-group, and $A$ a normal subgroup of order $p$. Show that $A \subset Z(P)$.

(10) Let $H$ be a normal subgroup of $G$ with $H$ having order $p$. Show that $H$ is contained in every $p$-Sylow subgroup of $G$.

(11) p. 151 ex. 6
(12) Let $A$ be a category, $X, Y \in \text{Ob}(A)$ and $\phi \in \text{Hom}(X, Y)$. For any $S \in \text{Ob}(A)$ consider the map of sets

$$\begin{align*}
\text{Hom}(S, X) &\xrightarrow{\phi^S} \text{Hom}(S, Y) \\
f &\mapsto \phi \circ f
\end{align*}$$

(a) Show that if $\phi$ is an isomorphism then $\phi^S$ is bijective.
(b) Suppose that $\phi^Y$ is bijective. Show that there is a map $g \in \text{Hom}(Y, X)$ such that $\phi \circ g = \text{id}_Y$. Conclude that $\phi = \phi \circ g \circ \phi$.  

1
(c) Suppose now both $\phi_*^X$ and $\phi_*^Y$ are bijective, and let $g$ be as above. Show that $g \circ \phi = id_X$. Conclude that $\phi$ is an isomorphism if and only if $\phi_*^S$ for all $S$.

(13) We say that a category $\mathcal{A}$ is pre-additive if for any $X, Y \in \text{Ob}(\mathcal{A})$ we are given an abelian group structure on $\text{Hom}(X, Y)$, such that composition

$$\text{Hom}(W, X) \rightarrow \text{Hom}(W, Y)$$

$$f \mapsto \phi \circ f$$

is a group homomorphism for any $\phi : X \rightarrow Y$, and similarly

$$\text{Hom}(X, Y) \rightarrow \text{Hom}(W, Y)$$

$$f \mapsto f \circ \phi$$

is a group homomorphism for any $\phi : W \rightarrow X$.

(a) Show that if $\mathcal{A}$ is a pre-additive category then $\mathcal{A}^{op}$ is also a pre-additive category with the same group structures.

(b) Show that the rule $(f + g)(x) := f(x) + g(x)$ gives the category $\mathcal{A}b$ of abelian groups the structure of a pre-additive category.

(14) Given a pre-additive category $\mathcal{A}$, objects $X, Y \in \text{Ob}(\mathcal{A})$ and given $f : X \rightarrow Y$, we say $\phi : K \rightarrow X$ is a kernel for $f$ if it satisfies the following universal property:

- $f \circ \phi = 0 \in \text{Hom}(K, Y)$.
- for any $\psi : S \rightarrow X$ such that $f \circ \psi = 0 \in \text{Hom}(S, Y)$, there is a unique $h : S \rightarrow K$ such that $\psi = \phi \circ h$.

(a) Consider $\mathcal{A} = \mathcal{A}b$. Show that if $f : X \rightarrow Y$ is an abelian group homomorphism, then the embedding $\text{Ker} f \hookrightarrow X$ is a kernel for $f$.

(b) What is a kernel for $f$ in $\mathcal{A}b^{op}$?