MATH 251 PROBLEMS

(1) Prove that the number of $p$-Sylow subgroups of $GL_2(\mathbb{F}_p)$ is $p+1$.

(2) Find a composition series for $A_4$.

(3) Describe the conjugacy classes in the dihedral group $D_{2n}$ and write the class formula explicitly.

(4) (a) Find a composition series for $GL_2(\mathbb{Z}/5\mathbb{Z})$

(b) Find a composition series for $GL_2(\mathbb{Z}/25\mathbb{Z})$

(to what extent is 5 important here?)

(5) Let $\mathcal{A}$ be a category, $X, Y \in Ob(\mathcal{A})$ and $\phi \in Hom(X,Y)$. For any $S \in Ob(\mathcal{A})$ consider the map of sets

\[ Hom(S, X) \xrightarrow{\phi_S} Hom(S, Y) \]

\[ f \mapsto \phi \circ f \]

(a) Show that if $\phi$ is an isomorphism then $\phi_S$ is bijective.

(b) Suppose that $\phi_X$ is bijective. Show that there is a map $g \in Hom(Y, X)$ such that $\phi \circ g = id_Y$. Conclude that $\phi = \phi \circ g \circ \phi$.

(c) Suppose now both $\phi_X$ and $\phi_Y$ are bijective, and let $g$ be as above. Show that $g \circ \phi = id_X$. Conclude that $\phi$ is an isomorphism if and only if $\phi_S$ for all $S$.

(6) prove that for an object $A \in Ob(\mathcal{C})$ the identity $id_A \in Hom(A, A)$ is unique.

(7) Show that for objects $S, T$ in a category, if $Isom(S, T)$ is nonempty then it is a principal homogeneous space for the group $Aut(T)$ (i.e. the group acts simply transitively).

(8) Complete the proof that for a fixed object $Y$, we have that $X \mapsto Hom(X,Y)$ is a contravariant functor and $X \mapsto Hom(Y,X)$ a covariant functor.

(9) Recall that a functor $F : A \rightarrow B$ is an equivalence of categories if it has a quasi inverse $G : B \rightarrow A$ such that the compositions are isomorphic to the identity functors. Prove that any two quasi inverses $G, G'$ of a functor $F$ are isomorphic.