Math 161 0 - Probability, Fall Semester 2012-2013
Dan Abramovich

Class meeting: Mondays, Wednesdays and Fridays at 11:00 am.

Dates: no class September 17, September 26.

Final exam December 21, 2012.

Midterm: October 24 in class.

I will hold class Sunday October 28 in the afternoon - details to come.

Classroom: Barus and Holley 163

Office hours: Mondays and Fridays at 10:00, Wednesdays at 2:00 pm.

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Text: Grinstead and Snell, Introduction to Probability.

This text is available free for download on
`http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/book.html`.

It is also available for purchase at the bookstore and on the American Mathematical Society web store:
`http://www.ams.org/bookstore-getitem/item=IPROB`
(last I checked it was on sale for $36).

Course outline: We'll cover most of the book.

Grading: there will be one midterm, one final, and weekly homework with possible projects.

Your final grade will be weighted as follows: Homework 30% Midterm 30% Final 40%.

Your lowest homework grade will be dropped.

Probability theory studies mathematical constructs called *probability spaces*. Typically these model results of experiments applied to natural or "real world" processes. You assume you know the underlying processes, and you try to predict the behavior of future experimental measurements: you have a coin, which you assume is fair, and you want to know the chances that in 50 throws it will fall on *heads* 45 times precisely.

The related discipline of *statistics* starts with measurements and aims to infer the underlying processes. You flipped a coin 100 times, it fell on *heads* 45 times precisely, and you want to know if it is more likely to be a fair coin or a biased coin with only 40% chances to fall on heads, and how confident you should feel about such likelihood.

Evidently, you must know your probability theory to be able to do statistics.

In real world applications, you must also know how to *model* the application as a probability space and *model* the measurment as a random variable.

Discrete probability spaces

Finite probability spaces

Example 1: you flip one fair coin once. The measurement is just Heads or Tails.

Probability space: (Ω, m)

$\Omega = \{H, T\}$ - the *sample space*

$m : \Omega \to [0, 1]$,

$m(H) = m(T) = 1/2$ - the probability *distribution function* or *measure* (Sometimes denoted P).

In this case the random variable X is just the value H or T, so no need to worry about it as yet.

(You could take it to be the identity function $X : \Omega \to \Omega$, which is what the book does.)

Example 2: you flip two fair coins. The measurement is the number of heads.

Probability space (Ω, m):

Sample space:

$\Omega = \{HH, HT, TH, TT\}$
$= \{H, T\} \times \{H, T\}$

Probabilities: $m : \Omega \to [0, 1]$,
$m(HH) = m(HT) = m(TH) = m(TT) = 1/4$

Random variable:

$X : \Omega \to \{0, 1, 2\}$:
$X(HH) = 2$;
$X(HT) = X(TH) = 1$;
$X(TT) = 0$.

We are interested in chances, or probabilities of *events*.

For instance: what's the probability that in example 2 the first coin falls as H?

$E_1 = \{HH, HT\}$;

$P(E_1) = m(HH) + m(HT)$

$= 1/4 + 1/4 = 1/2.$

What's the probability that the number of heads is 2?

$E_2 = \{\omega \in \Omega : X(\omega) = 2\}$

$= \{HH\}$;

$P(E_2) = m(HH) = 1/4$

Definition: a *finite probability space* is a pair (Ω, m) where Ω is a finite set - called *sample space*, $m : \Omega \to [0,1]$ is a function, called *probability measure*, or *distribution function*, such that

$$\sum_{\omega \in \Omega} m(\omega) = 1.$$

An *event* is a subset $E \subset \Omega$. The *probaility of E* is

$$P(E) = \sum_{\omega \in E} m(\omega).$$

A *random variable* is a function $X : \Omega \to S$ to some other set S.

Two cases are often considered: in the book, $X : \Omega \to \Omega$ the identity function is featured first. In most later applications one consideres $S = \mathbb{R}$, the real numbers, and $X : \Omega \to \mathbb{R}$ is a *real valued random variable.*

Example 2, revised: The book's description is not outlandish. You could take instead (Ω', m') with
$\Omega' = \{0, 1, 2\} \subset \mathbb{R}$
and
$m'(0) = 1/4, m'(1) = 1/2, m'(2) = 1/4.$

Then X is again the identity function. But we will not worry about this duality of meaning till much later.

Example 3 you throw 2 fair dice. What's the probability that the sum of the numbers is 3?

model:

$\Omega = \{(i, j) : i, j = 1, \ldots, 6\}$.

$m(i, j) = 1/36$

uniform distribution.

$X : \Omega \to \mathbb{R}; X((i, j)) = i + j$

$E = \{X = 3\} = \{(1, 2), (2, 1)\}$.

$P(E) = 2/36$.

Countable probability spaces

We can do the same with a countably infinite space. No way we can have it uniform!

Definitions are the same, with the assumption that the sum $\sum_\omega m(\omega)$ converges and equals 1.

Example We throw a fair coin until heads appears, and record the number of throws. (If heads never appears record ∞.) What's the probability that we stop after an odd number of times?

$\Omega = \{1, 2, 3. \ldots\} \cup \{\infty\}$.

$m(n) = 2^{-n}$, since you need tails to appear $n - 1$ times, and then heads to appear (probability 1/2 each).

Still need $\sum_{\omega\in\Omega} m(\omega) = 1$. So $m(\infty) = 1 - \sum_{n=1}^{\infty} m(n)$. But

$$\sum m(n) = \sum_{n=1}^{\infty} 2^{-n} = \frac{1/2}{1-1/2} = 1,$$

so $m(\infty) = 0$.

(We could remove ∞ for all practical purposes.)

$E = \{n \text{ is odd}\}$

$$p(E) = \sum_{n \text{ odd}} 2^{-n} = \frac{1/2}{1-1/4} = 2/3.$$

Challenge: play the same with a die, till 1 appears. Describe the probability space. What's the probability that you stop no earlier than 5 steps? after an odd number of steps?

Theoem.

(1) $0 \leq P(E) \leq 1$.
(2) $P(\{\omega\}) = m(\omega)$.
(3) $P(\Omega) = 1$
(4) $E \subset F \subset \Omega \Rightarrow P(E) \leq P(F)$.
(5) $A \cap B = \emptyset \Rightarrow P(A \cup B) = P(A) + P(B)$.
(6) $P(\Omega - E) = 1 - P(E)$.

(1-3) follow from the definition, and (4) and (6) follow from (5).

Proof of (5):

$P(A \cup B) = \sum_{\omega \in A \cup B} m(\omega)$
$= \sum_{\omega \in A} m(\omega) + \sum_{\omega \in B} m(\omega)$
$= P(A) + P(B)$,

since $A \cap B = \emptyset$.

(5) generalizes to arbitrarily many (or infinitely many) disjoint events.

Corollary.

If $\Omega = \sqcup A_i$ then $P(E) = \sum P(E \cap A_i)$

Proof: $E_i = E \cup A_i$ are disjoint with union E.

If there is an intersection we can still break things down:

Theorem.

$P(A \cup B)$
$= P(A) + P(B) - P(A \cap B)$.

Challenge: generalize to many events!

Remark about odds:

When you say there are $r : s = 15 : 100$ odds that Bucky will win the horse race, what does it mean in terms of probabilities?

We imagine a total of $15+100$ races out of which Bucky wins 15.

So $p = P(E) = P(\text{Bucky wins}) = \frac{15}{15+100}$.

In general: $p = \frac{r}{r+s}$.

to reverse, just solve:

$p = \frac{r/s}{(r/s)+1}$

so $r/s \ = \ p/(1-p)$.