Dan Abramovich

Continuous probabilities

Example 1: Consider a spinner. You assume the angle is "random" and record
$X = \alpha/(2\pi) \in [0, 1)$.

What's the probability that $X = 0$?

What's the probability that the spinner's head lands in the top half, i.e. $X < 1/2$?

Example 2: consider the square
$\{(x, y) : 0 \leq x, y \leq 1\} = [0, 1]\times[0, 1]$.
Assume the coordinates are chosen "at random". What's the probability that $y \leq x^3$?
What's the probability that
$x^2 + y^2 < 1$?

In these examples we had Ω a set of cardinality a continuum. It makes no sense to require a positive probability for every $\omega \in \Omega$.

In this course we will allow no positive probability for any $\omega \in \Omega$.

What's important in this case is that there are *measurable events* for which we might expect a positive probability!

The rules of the game are spelled out in our countable theorem:

(1) $P(\emptyset) = 0$; $P(\Omega) = 1$.

(2) $0 \leq P(E) \leq 1$

(3) If A_i are disjoint measurable events, then $P(\cup A_i) = \sum_i P(A_i)$.

at least for finitely many A_i

One case when this happens:

$\Omega =$ interval,

Measurable events: intervals, or finite unions of intervals.

And then set

$$P(E) = \int_E f(t)\, dt.$$

What do we need from such $f(t)$?

(1) Riemann integrable.

(2) $f(t) \geq 0$

(3) $\int_\Omega f(t)\, dt = 1$

Uniform distribution: $f = 1/Vol(\Omega)$ constant!

Another case: $\Omega \subset \mathbb{R}^n$ a reasonable domain, and

$$P(E) = \underset{E}{\int \cdots \int} f(x_1, \ldots, x_n)\, dx_1 \cdots dx_n.$$

The function f has to satisfy precisely the same axioms as before.

For instance, when you play darts on $\Omega = \{x^2 + y^2 \leq 1\}$, the uniform choice of f would be $f = 1/\pi$, so

$$P(E) = \iint_E \frac{1}{\pi}\, dx\, dy.$$

Such a function is called a **probability density function**, and the pair

(Ω, f) will constitute a probability space, though this is not the standard terminology.

If $X : \Omega \to \mathbb{R}$ is a continuous function it will be a called a *random variable*.

Even if Ω is complicated, one can get simplified pictures using random variables. But one needs some definitions and care as we'll see.

Cumulative distribution function:

$$F_X(x) = P(X \le x).$$

If $\Omega = \mathbb{R}$ and $X(\omega) = \omega$ we have the following relationship by the FTC:

Theorem.

$$F(x) := F_X(x) = \int_{-\infty}^{x} f(t)\, dt,$$

and

$$\frac{d}{dt} F(x) = f(x)$$

whenever defined.

Note: $\Omega = [0,1]$ with uniform density $f = 1$. Let $X(t) = t^2$. What is the cumulative distribution?

Assume $0 \le x \le 1$. Then

$F_X(x) = P(X \le x) = P(t^2 \le x)$

$= P(t \le \sqrt{x}) = \sqrt{x}$.

This defines $F_X(x) = \sqrt{x}$ for $0 \le x \le 1$. We extend it for all x by noting that $P(X < 0) = 0$ and $P(X > 1) = 0$, so

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ \sqrt{x} & 0 \le x \le 1 \\ 1 & x \ge 1 \end{cases}$$

The function $f_X(x) = \frac{d}{dt}F_X(x)$ is a density function for $\mathbb{R}$ as a new probability space measuring the probabilities of events in terms of X. So

$$P(X < x) = \int_{-\infty}^{x} f_X(x)\,dx.$$

Note that in the last example $f_X(x)$ is not continuous.

Challenge: do the same with the random variable $Y = -\log t$.

Example Alice and Bob arrive at BMC between 10 and 11. Assume each arrives independently "at random". What's the cumulative distribution function of the time the first has to wait for the second?

$\Omega = [0,1]^2$, A = arrival of Alice; B = arrival of Bob. Wish to study $X = |A - B|$.

$F_X(x) = P(|A-B| < x) = P(E_x)$

$P(E_x) = 1-(1-x)^2$ for $0 \le x \le 1$

What's the density?

$$f_X(x) = \begin{cases} 0 & x < 0 \\ 2 - 2x & 0 < x \le 1 \\ 0 & 1 < x \end{cases}$$

Exponential distributions

$$f_\lambda(t) = \begin{cases} \lambda e^{-\lambda t} & t > 0 \\ 0 & t < 0 \end{cases}$$

Claim: this is a density function.

Challenge: What's the cumulative distribution?

As we'll see this is a model for waiting times for random occurrences (an inconsiderate bus company).

It is "memory-less" - we'll come back to this when we talk about conditional probabilities.

Beautiful example: simulating contnuum by infinity of discrete variables. Countably many fair coin tosses. H = 1, T= 0.

$\Omega = \{0,1\}^{\mathbb{N}}$, probability as "independent variables".

$B_n : \Omega \to \{0,1\}$ the n-th coin toss

$X = \sum B_n/2^n : \Omega \to \mathbb{R}$

What is $F_X(0.1_2)$?

This is $P(B_1 = 0) = 1/2$

What is $F_X(0.01_2)$?

This is $P(B_1 = B_2 = 0) = 1/4$

What is $F_X(0.11_2)$?

This is $P(B_1 = 0 \text{ or } B_2 = 0) = 3/4$

Inductively, if x is a finite binary number then $F_X(x) = x$, and by continuity this will hold for every x!

Note: $f_X(x) = F'_X(x)$.

So $F_X(x)$ is a uniform distribution!