Permutations

How many pernutations of $1, \ldots, n$?

$n! = n \cdot (n-1) \cdots 2 \cdot 1.$

k-permutations: selections of an ordered k tuple of distinct elements of $1, \ldots, n$. How many?

$(n)_k =$
$= n \cdot (n-1) \cdots (n-k+1)$
$= \frac{n!}{(n-k)!}.$

Same with $\{1, \ldots, n\}$ replaced by any set of size n.

How large is $n!$?

Very large! Stirling's formula:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

Challenge: Show at least that

$$n! \Bigg/ \left(\left(\frac{n}{e}\right)^n \sqrt{n}\right)$$

and its reciprocal remain bounded.

(Ask for hints after you struggle a bit!)

Later on we'll be able to study questions like:

What's the statistics of the number of fixed points of a permutation?

What's the statistics of the number of records in a finite sequence of results?

Combinations

How many ways to choose a subset of $\{1, \ldots, n\}$ of size k?

$$\binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}.$$

Name: n choose k, binomial coefficient.

Four of a kind beats full house:

$\#$ Four of a kind $= 13 \times 48 = 624$.

$\#$ full houses
$= (13 \times \binom{4}{3}) \times (12 \times \binom{4}{2}) = 3744$
$= 624 \times 6$

Relationships:

$$\binom{n}{k} = \binom{n}{n-k}.$$

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

$$\sum_{k=0}^{n} \binom{n}{k} \quad = \quad 2^n.$$

$$\sum_{k \text{ even}} \binom{n}{k} \quad = \quad 2^{n-1}.$$

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}.$$

Bernoulli trials.

This is one of the most common probability spaces.

You perform n identical experiments, without one result affecting the other. Each experiment has two outcomes: either success, with probability p, or failure, with probability $1 - p$.

$$\Omega = \left\{ \begin{matrix} \text{ordered sequences of "S"} \\ \text{and "F" of length } n \end{matrix} \right\}$$

Say $\omega \in \Omega$ has k "S" and $n - k$ "F".

$m(\omega) = p^k (1 - p)^{n-k}$.

Examples: coins (possibly biased)

modeling a survey

gambling with a small bias for failure

modeling medical trials

Binomial probabilities

This is one of the most common measurements on bernoulli trials.

Say $X(\omega) = \#$ of "S".

What is the probability that $X = k$?

$b(n, p, k) := P(X = k)$

$$b(n, p, k) = \binom{n}{k} p^k (1-p)^{n-k}.$$

This is a probability distribution on $\{0, \ldots, n\}$, the *binomial distribution*.

It has a familiar bell-shaped "curve".

(Breaking it down:

Say $X_i(\omega) = \begin{cases} 1 & \omega_i = S \\ 0 & \omega_i = F \end{cases}$

Then $X = \sum X_i$. We'll use this later.)

Inclusion-Exclusion

Theorem.

$$
\begin{aligned}
&P(A_1 \cup \cdots \cup A_n) \\
&= \sum_i P(A_i) \\
&- \sum_{i<j} P(A_i \cap A_j) \\
&+ \sum_{i<j<k} P(A_i \cap A_j \cap A_k) \\
&- \cdots .
\end{aligned}
$$

Proof. On the left you get a sum of $m(\omega)$, where ω counts if it is in at least one of the A_i.

How many times is it counted on the right? if ω is in precisely $k > 0$ of the A_i, we count $m(\omega)$ precisely $\binom{k}{1} - \binom{k}{2} + \cdots \pm \binom{k}{k}$ times. But $1 - (\binom{k}{1} - \binom{k}{2} + \cdots \pm \binom{k}{k}) = (1 - 1)^k = 0$, so it is counted once! ♣

Remark Later on we'll discuss characteristic functions, which make this a bit more comfortable.

Challenge: Think about other methods of proof!

Fixed points of a permutation: the checkroom attendant returns n people's distinct hats at random. What's the probability *nobody* gets his/her own hat?

Surprisingly, it is more convenient to look at the probability of at least one matched hat! (We'll see later that even easier it is to calculate the *expected number* of matches.)

A_i = event that i-th hat goes to the i-th person

We are looking for $P(A_1 \cup \cdots \cup A_n)$.

Need:

$P(A_{i_1} \cap \cdots \cap A_{i_k})$ for $i_1 < \cdots < i_k$.

Now by the uniformity assumption,

$$P(A_{i_1} \cap \cdots \cap A_{i_k}) = \frac{(n-k)!}{n!} = \frac{1}{(n)_k}.$$

So the k-th term is

$$\sum P(A_{i_1} \cap \cdots \cap A_{i_k}) = \binom{n}{k} \Big/ (n)_k = \frac{1}{k!}.$$

So

$$P(A_1 \cup \cdots \cup A_n) = \frac{1}{1!} - \frac{1}{2!} + \cdots + (-1)^{n-1} \frac{1}{n!},$$

and

$$P(\text{no fixed point}) = 1 - \frac{1}{1!} + \frac{1}{2!} - \cdots + (-1)^n \frac{1}{n!}.$$

This approaches e^{-1} very fast!