Math 161 0 - Probability, Fall Semester 2012-2013
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Conditional probabilities

$P(F|E) \quad := \quad P(F \cap E)/P(E).$

Makes sense only if $P(E) \neq 0$.

Discrete distribution function: $m(\omega) = P(\{\omega\})$, so

$$m(\omega|E) \quad := \quad \begin{cases} \frac{m(\omega)}{P(E)} & \omega \in E \\ 0 & \omega \notin E. \end{cases}$$

The effect is to replace Ω by E!

Continuous:

$$f(\omega|E) \quad := \quad \begin{cases} \frac{f(\omega)}{P(E)} & \omega \in E \\ 0 & \omega \notin E. \end{cases}$$

Example 1: cast a die. $E = \{\omega \geq 4\}$, $F = \{\omega = 6\}$.

$P(F|E) = (1/6)/(1/2) = 1/3$.

$P(E|F) = 1$.

Example 2: two urns, 2B+3W, 1B+1W

$P(I|B) =?$

$(2/10)/(2/10 + 1/4)$

"Bayes probabilities"

Example: Monty Hall problem.

Say player chooses door 1 always (see book for removing this)

draw table 13: 1/6; 12:1/6

23: 1/3

32: 1/6

$P(C \neq 1|M = 3) = (1/3)/(1/3 + 1/6) = 2/3.$

cgG	1/6	cGg	1/6
gcG	1/3	gCg	0
ggC	0	gGc	1/3

independent events

$P(F|E) = P(F)$

Claim: if nonzero, implies $P(F|E) = P(F)$

Proof: the definition says $P(F|E) = P(E \cap F)/P(E)$ so $P(F|E)P(E) = P(E \cap F)$.

Now $P(E|F) = P(E \cap F)/P(F) = P(F|E)P(E)/P(F) = P(E)$ by assumption.

Can do the same for a collection of events: A_i are independent if

$P(A_{i_1} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdots P(A_{i_k})$ for distinct i_j.

Example: A_1 first coin heads, A_2 second coin heads, B =even number of heads.

independence and random variables

Take $X_i : \Omega \to R_i$
random variables, $i = 1, \ldots, n$.

Can think of R_i as new sample space:

$$m(r) := P(X = r).$$

Also can think of

$$\Omega' = R_1 \times R_2 \cdots \times R_n$$

as a sample space:

$$m(r_1, \ldots, r_n) :=$$
$$P(X_1 = r_1, \ldots, X_n = r_n).$$

"Joint distribution"
Pretend $X_i : \Omega' \to R_i$
Definition: X_i are independent if the events $X_i = r_i$ are.

Independent trials process (IID random variables)

Think of X_i being the results of different throws of fair dice. Then $m(r_1, \ldots, r_n) = m(r_1) \cdots m(r_n)$.

Definition: independent trials process: all X_i have the same distribution on the same sample space R_i, and are independent.

Bayes formula

We have a population of people. We want to find out what type of person fell into our clinic.

We have done an experiment and discovered evidence $E =$ the patient has the condition "buzz haircut".

We have data on diffferent types of people and want to check how likely possible hypotheses are.

Know: probability of E given any one of the hypotheses.

$$P(H_i|E) = \frac{P(E|H_i)P(H_i)}{\sum_j P(E|H_j)P(H_j)}$$

$\Omega = H_1 \cup H_2 \cup H_3 \cup H_4$

H_1 : female child H_2 : female adult H_3 : male child H_4 : male adult

Want: $P(H_i|E)$.

From previous surveys, our statistician has collected the probabilities $P(E|H_i)$ and $P(H_i)$.

$P(H_i|E) = P(H_i \cap E) \ / \ P(E)$.

But $P(E) = P(E \cap H_1) + \cdots + P(E \cap H_4)$

and $P(E \cap H_i) = P(E|H_i)P(H_i)$.

so

$$P(H_i|E) = \frac{P(E|H_i)P(H_i)}{\sum_j P(E|H_j)P(H_j)}$$

Say

$P(E|H_1) = 0; P(H_1) = .15$

$P(E|H_2) = .1; P(H_2) = .35$

$P(E|H_3) = 0.4; P(H_3) = .15$

$P(E|H_4) = 0.6; P(H_4) = .35$

$$P(H_3|E) = \frac{P(E|H_3)P(H_3)}{\sum_j P(E|H_j)P(H_j)}$$

$= \frac{0.4 \times 0.15}{0 \times .15 + .1 \times .35 + 04 \times .15 + .6 \times .35}$

$= \frac{0.06}{0 + 0.015 + 0.06 + 0.21}$

$= 60/285 \sim 0.21$

Assume one in 1000 has skin cancer

Assume test is positive or negative according to the following conditional probabilities

	+	-
C:	.99	0.1
H:	0.1	.99

What is $P(C|-)$?

$$P(C|+) = \frac{P(+|C)P(C)}{P(+|C)P(C) + P(+|H)P(H)}$$

$$= \frac{0.99 \times 0.001}{0.99 \times 0.001 + 0.01 \times 0.999}$$

$= 0.00099/(0.00099 + 0.00999)$

$= 99/1098 \sim .09$

Very important for $P(P|H)$ - the probability of a false positive - to be low.