Math 161 0 - Probability, Fall Semester 2012-2013
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Continuous conditional probabilities Still the case that

$P(F|E) = P(F \cap E)/P(E)$

Book example: suppose we know the dart landed on the top part. What is the probability that it is at distance 1/2 from center?

Consequence:

$$f(\omega|E) \quad := \quad \begin{cases} \frac{f(\omega)}{P(E)} & \omega \in E \\ 0 & \omega \notin E. \end{cases}$$

Exponential

$f(x) = \lambda e^{-\lambda t}$ for some $\lambda > 0$.

if $E = \{x > t\}$, then

$$P(E) = \int_t^\infty \lambda e^{-\lambda t} dt = -e^{-\lambda t}\Big|_t^\infty = e^{-\lambda t}.$$

If now $F = \{x > t+s\}$ then $P(F \cap E) = P(F) = e^{-\lambda(t+s)}$. So

$$P(F|E) = e^{-\lambda(t+s)}/e^{-\lambda t} = e^{-\lambda s}.$$

no memory!

Good for bulbs, radioactive emissions. Hopefully not for your bus.

$$F(x_1, \ldots x_n) = P(\forall i, X_i \leq x_i).$$

So

$$F(x_1, \ldots x_n) = \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_n} f(t_1, \ldots, t_n) dt_1 \cdots dt_n$$

$$f(t_1, \ldots, t_n) = \frac{\partial^n F(x_1, \ldots x_n)}{\partial x_1 \cdots \partial x_n}.$$

The random variables are **independent** if

$F(x_1, \ldots x_n) = F(x_1) \cdots F(x_n)$.

Equivalently

$f(x_1, \ldots x_n) = f(x_1) \cdots f(x_n)$.

Examples: the randon variables x, y on the uniform unit square are independent. But x and $x + y$ are not. But x and $x + y \mod 1$ are.

Theorem: x_i are independent, and ϕ_i are functions, then $\phi_i(x_i)$ are independent.

Towards beta densities

Experiment: we have a drug which is effective with probability x.

Unfortunately we do not know what x is. We make the completely uninformed assumption that x is uniformly distributed on $[0, 1]$.

We try it on n people and get i successes.

Draw the sample space - an interval for each $0 \leq i \leq n$.

Key question: what is the probability of success in the next trial, conditional on this?

Answer:

$$P(S|i \text{ S's out of } n) = \frac{i+1}{n+2}.$$

What we surely know:

$m(i|x) = \binom{n}{i} x^i (1-x)^{n-i}$.

We have not put this into our formalism, but this does make sense. So

$f(x,i) = \binom{n}{i} x^i (1-x)^{n-i}$.

So

$m(i) = \int_0^1 \binom{n}{i} x^i (1-x)^{n-i} dx = ?$

Fact: for integers i, j we have

$$\int_0^1 x^i(1-x)^j dx = \frac{i! \cdot j!}{(i+j+1)!}$$

$=: B(i+1, j+1)$.

One in fact *defines*

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx$$

which enables you to define factorials for any real α.

It follows that
$f(x|i) = f(x,i)/m(i) = \frac{x^i(1-x)^{n-i}}{B(i+1,j+1)}$
with $j = n - i$.

So *after* i *successes in* n *trials* this gives us a new starting distribution for $0 \le x \le 1$:

Denote

$$B(\alpha, \beta, x) = \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}.$$

So $f(x|i) = B(i+1, j+1, x)$.

If the experiment now goes for the $n+1$ trial, then

So
$f(S|i) = \int_0^1 B(i+1, j+1, x) \cdot x dx$
$= \frac{B(i+2,j+1)}{B(i+1,j+1)} = \frac{i+1}{n+2}$

Paradox of the envelopes

You are given two envelopes, with different positive amounts of money in whole dollars. You choose one at random between the two.

You are not told how the amounts were chosen. We assume that it is drawn from some distribution. If x is the amount in the envelope you chose and y is the other, then you can assume there is some distribution $m(x, y)$ on

$$\Omega = \{(x, y) \in \mathbb{N}^2 | x \neq y\},$$

such that

$$\forall x, y, \; m(x, y) = m(y, x)$$

Now we take an independent variable Z on $\mathbb{N}$ with the property that $\forall z,\ m(z) \neq 0$.

This gives a new sample space $\Omega \times \mathbb{N}$ such that

$$m(x, y, z) = m(x, y) \cdot m(z).$$

We make the rule that you switch the envelope if and only if $z \geq x$.

Claim: with probability $> 1/2$ we chose the bigger amount.

Proof: The sample space is split into disjoint events:

$$\begin{aligned}
XYZ &= \{x < y \le z\} && \text{(good switch, win)}\\
YXZ &= \{y < x \le z\} && \text{(bad switch, lose)}\\
XZY &= \{x \le z < y\} && \text{(good switch, win)}\\
YZX &= \{y \le z < x\} && \text{(no switch, win)}\\
ZXY &= \{z < x < y\} && \text{(no switch, lose)}\\
ZYX &= \{z < y < x\} && \text{(no switch, win)}
\end{aligned}$$

Note that $P(XYZ) = P(YXZ)$ and $P(ZXY) = P(ZYX)$.

So $P(\mathrm{w}) - P(\mathrm{l}) = P(XZY) + P(YZX)$. By our assumption the latter is positive.