Math 161 0 - Probability, Fall Semester 2012-2013
Dan Abramovich

Selected distributions

Geometric distribution

$T =$ number of independent bernoulli trials of prob p till first success:

$P(T = n) = q^{n-1}p.$

Have seen: adds up to 1.

Memoryless:

$P(T = n+k | T > n) = q^{n+k-1}p/q^n$
$= q^{k-1}p = P(T = k)$

Variant: "Negative binomial distribution"

Independent bernoulli-p-trials till we get k successes. $X =$ time.

$u(x, k, p) := P(X = x) = \binom{x-1}{k-1} p^k q^{x-k}$

Poissson distribution

This is a good way to approximate binomial distributions, since those are usually a lot of work to sum up.

If you do $n = 100$ trials, one per hour, with probability p, you expect roughly the same number of successes as doing $n = 200$ trials, two per hour, but with probability $p/2$. These converge rather rapidly to the following limit:

Write $\lambda = np$. Think of λ as the average number of successes per hour.

So $p = \lambda/n$. As $n \longrightarrow \infty$ we have $p \longrightarrow 0$, so

$b(n,p,k) = \binom{n}{k} p^k q^{n-k}$

$= \frac{(n)_k p^k}{k!} \frac{\left((1-p)^{(1/p)}\right)^{\lambda}}{(1-\lambda/n)^k} \underset{n\to\infty}{\longrightarrow} \frac{\lambda^k}{k!} e^{-\lambda}$

We set

$$\text{Poisson}(\lambda, k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

hypergeometric distribution. Urn has 100 balls, 40 blue balls and the rest red balls. What's the probability of getting 5 blue balls and the rest red balls, drawing 20 balls at random?

This is just counting:

$$h(100, 40, 20, 5) = \frac{\binom{40}{5}\binom{100-40}{20-5}}{\binom{100}{20}}.$$

In general

$$h(N, k; n, x) = \frac{\binom{k}{x}\binom{N-k}{n-x}}{\binom{N}{n}}.$$

Read about benford

$f(k) = \log_{10}(k+1) - \log_{10}(k)$.

Comes from taking $\log x \mod 1$.

Densities
Uniform

on an interval $[a, b]$ set
$$f(x) = 1/(b - a).$$

Exponential

on $x \geq 0$ set $f(x) = \lambda e^{-\lambda x}$.

We have calculated $F_X(x) = 1 - e^{-\lambda x}$. The more natural finction is $G(x) = P(X > x) = e^{-\lambda x}$.

We have seen it is memoryless.

Note that if we set $k = \lfloor x \rfloor$, then k is geometric with $p = 1 - e^{-\lambda}$. This explains the memoryless property of geometric distributions.

Functions of random variables

Theorem: X continuous random variable, $\phi(x)$ a *strictly increasing* function. Write $Y = \phi(X)$. Then

$$F_Y(y) = F_X(\phi^{-1}(y).$$

$$F_Y(\phi(x)) = F_X(x).$$

If ψ strictly decreasing and $Z = \psi(x)$ then

$$F_Z(z) = 1 - F_X(\psi^{-1}(z).$$

Proof: write out events:

$$\begin{aligned} F_Y(y) &= P(Y \le y) \\ &= P(\phi(X) \le y) = P(X \le \phi^{-1}(y)) \\ &= F_X(\phi^{-1}(y)) \end{aligned}$$

Similar for ψ!

Corollary

$$f_Y(y) = f_X(\phi^{-1}(y)) \cdot \frac{d}{dy}\phi^{-1}(y)$$

$$f_Z(z) = -f_X(\psi^{-1}(z)) \cdot \frac{d}{dz}\psi^{-1}(z)$$

What if $W = |X|$?

$$F_W(w) = F_X(x) - F_X(-x)$$

$$f_W(w) = (f_X(x) + f_X(-x)).$$

Normal density

$X \sim Normal(\mu, \sigma)$

$$f_X(x) \quad := \quad \frac{1}{\sqrt{2\pi}\sigma} \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, .$$

Centered around μ, width regulated by σ.

$$F_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt.$$

```
In[6]:= 1/Sqrt[2*Pi]
*Integrate[Exp[-x^2/2],{x,-1,1}]

                    1
Out[6]= Erf[-------]
                Sqrt[2]
```

Note: if $Z \sim Normal(0,1)$ then $X = \sigma Z + \mu$.

$$F_X(x) = F_Z\left(\frac{x-\mu}{\sigma}\right)$$

indeed

$$f_X(x) = f_Z\left(\frac{x-\mu}{\sigma}\right)/\sigma$$

$$= \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

READ REST OF LAUNDRY LIST